

A STRONGLY ILL-POSED PROBLEM FOR A DEGENERATE PARABOLIC EQUATION WITH UNBOUNDED COEFFICIENTS IN AN UNBOUNDED DOMAIN $\Omega \times \mathcal{O}$ OF \mathbb{R}^{M+N}

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ABSTRACT. In this paper we deal with a strongly ill-posed second-order degenerate parabolic problem in the unbounded open set $\Omega \times \mathcal{O} \subset \mathbb{R}^{M+N}$, related to a linear equation with unbounded coefficients, with no initial condition, but endowed with the usual Dirichlet condition on $(0, T) \times \partial(\Omega \times \mathcal{O})$ and an additional condition involving the x -normal derivative on $\Gamma \times \mathcal{O}$, Γ being an open subset of Ω .

The task of this paper is twofold: determining sufficient conditions on our data implying the uniqueness of the solution u to the boundary value problem as well as determining a pair of metrics with respect of which u depends continuously on the data.

The results obtained for the parabolic problem are then applied to a similar problem for a convolution integrodifferential linear parabolic equation.

1. INTRODUCTION

In the second half of the last century a lot of interest, due to the rushing on of Technology, was devoted to Inverse Problems, a branch of which consists just of strongly ill-posed problems, where *strongly* means that no transformation can be found in order to change such problems to well-posed ones, at least, say, when working in classical or Sobolev function spaces of *finite order*.

Assume that you are dealing with the evolution of the temperature u involving a body ω occupying a (possibly) unbounded domain in \mathbb{R}^{M+N} , and assume that you cannot measure the temperature u inside ω , but you can perform only measurements on the boundary of ω . So, you have no initial condition at your disposal, but only several boundary measurements of temperature, flux and so on. This makes the parabolic problem strongly ill-posed. The basic questions which arise in this case are the following:

- (i) may the solution to this problem be unique?
- (ii) in this case may the solution depend continuously on the boundary data?
- (iii) if this is possible, which are the allowed metrics?

This paper is devoted to shed some light on degenerate parabolic problems of that kind on (possibly) unbounded domains $\omega = \Omega \times \mathcal{O}$, where $\Omega \subset \mathbb{R}^M$ and $\mathcal{O} \subset \mathbb{R}^N$ are two smooth open sets, the first being bounded, while the latter is unbounded. More precisely, we consider operators \mathcal{A} , defined on smooth functions $\zeta : \Omega \times \mathcal{O} \rightarrow \mathbb{R}$ by

$$\begin{aligned} \mathcal{A}\zeta(x, y) = & \operatorname{div}_x(a(x)\nabla_x\zeta(x, y)) + \sum_{i=1}^M c_i(x, y)D_{x_i}\zeta(x, y) \\ & + \sum_{j=1}^N b_j(y)D_{y_j}\zeta(x, y) + b_0(x, y)\zeta(x, y), \end{aligned}$$

for any $(x, y) \in \Omega \times \mathcal{O}$. We assume that the function a nowhere vanishes in $\overline{\Omega}$. Anyway operator \mathcal{A} is degenerate since its leading part contains second-order derivatives computed only with respect to the variables x_1, \dots, x_N .

We will be concerned mainly with the questions of uniqueness and continuous dependence on the data (two fundamental topics for people working in Applied Mathematics) of the nonhomogeneous linear parabolic equation associated with the operator \mathcal{A} in $(0, T) \times \Omega \times \mathcal{O}$, with no initial conditions. The lack of the initial conditions is replaced by the requirement that the “temperature” u should assume prescribed values on $(0, T) \times \partial(\Omega \times \mathcal{O})$, while the x -normal derivative of u should assume prescribed values on an open subsurface $(0, T) \times \Gamma \times \mathcal{O}$ of the lateral boundary $(0, T) \times \partial\Omega \times \mathcal{O}$.

The fundamental tool to give some positive answer to our problem are new Carleman estimates that fits our case. Following the ideas in [17, Theorem 3.4], we will construct suitable Carleman inequalities related to an unbounded open set.

We then show that our technique can be adapted to deal also with some degenerate integrodifferential parabolic boundary value problems and with some class of degenerate semilinear boundary value problems.

Carleman estimates, entering many applications in Control theory (see e.g., [22, 34]) and in unique continuation theorems (see e.g., [26]) have shown to be a powerful tool in studying inverse and ill-posed problems for partial differential equations. Starting from the pioneering works in the eighties by Bukhgeim and Klibanov (see [8, 27, 28] and also the monographs [7, 30] and the survey papers [25, 29]), Carleman estimates have been used to solve identification problems, mainly in bounded domains, associated with nondegenerate differential operators. We quote, e.g., [3, 4, 6, 5, 18, 24, 33, 39]. On the contrary, to the best of our knowledge, Carleman estimates have not been extensively used so far in the analysis of inverse problems in unbounded domains. We are aware only of the papers [15, 16]. In [15] Carleman estimates have been used to uniquely recover the unknown function c in a Cauchy problem for the Schrödinger equation

$$i \frac{\partial q}{\partial t} + \operatorname{div}(c \nabla q) = 0,$$

related to a strip of \mathbb{R}^2 , from the knowledge of the normal derivative of the time derivative of q on the upper boundary of the strip. More recently, in [16] the authors have considered the more general form of the Schrödinger equation

$$i \frac{\partial q}{\partial t} + a \Delta q + b q = 0,$$

and they have shown that the knowledge of the normal derivative of the second-order time derivative of the solution on the same part of the boundary of the strip as in [15], allows for recovering the two functions a and b . Both in [15] and in [16] a nondegeneracy condition is assumed on the elliptic part of the operator. Moreover, the coefficients are assumed to be at least bounded.

Similarly, Carleman estimates for degenerate parabolic problems seem to have not been so far widely used to solve inverse problems. We are aware of the papers [14, 35, 36, 37]. In [14, 35] Carleman estimates are used to recover the unknown function g entering the degenerate one-dimensional heat equation

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x} \left(x^\alpha \frac{\partial u}{\partial x} \right) = g,$$

related to the spatial domain $(0, 1)$, and where $\alpha \in [0, 2)$.

In [36, 37] such estimates are used to solve an identification problem for a boundary value problem associated with the heat equation

$$\frac{\partial u}{\partial t} = \Delta u + \frac{\mu}{|x|^2} u + g$$

in a boundary open set containing 0, with no initial condition and μ is a positive constant not larger than the optimal constant in Hardy's inequality. The Carleman estimates obtained by the author extends similar estimates obtained in [20, 38].

On the other hand, Carleman estimates for degenerate parabolic equations have been more widely used in Control Theory, but mainly associated to one-dimensional parabolic operators (we quote e.g., [1, 9, 10, 11, 12, 13, 21] and the reference therein).

At present, we are not aware of other papers where Carleman estimates are proved for *degenerate* parabolic operators with *unbounded* coefficients, which are related to an *unbounded* spatial domain.

The plan of the paper is the following: in Section 2 we exactly state the ill-posed degenerate differential problem, while in Section 3 we prove two theorems involving Carleman estimates for our problem, implying the uniqueness of our solution. Section 4 is devoted to finding a continuous dependence result for the solution to our problem in the usual space $L^2(\Omega \times \mathcal{O})$. Finally, in Section 5 we extend our results to both to a convolution integrodifferential equation (see Subsection 5.1) and to a class of semilinear equations (see Subsection 5.2).

Notations. Throughout the paper we denote by $\|f\|_\infty$ the sup-norm of a given bounded function f . If $f \in C^k(\overline{\Omega})$ for some $k \in \mathbb{N}$ and some bounded domain $\Omega \subset \mathbb{R}^M$, we denote by $\|f\|_{k,\infty}$ the Euclidean norm of f , i.e., $\|f\|_{k,\infty} = \sum_{|\alpha| \leq k} \|D^\alpha f\|_\infty$. The same notation is used to denote the $W^{k,\infty}$ -norm of a function a .

Given a square matrix B , we denote by $\|B\|$ its Euclidean norm.

Typically, the function spaces that we consider consist of real-valued functions but in Sections 2 and 4, and in the first part of Section 3, where we need complex-valued functions for our integrodifferential application. In this case we use the subscript “ \mathbb{C} ” to denote function spaces consisting of complex-valued functions.

The inner product in \mathbb{R}^K will be denoted by “ \cdot ”. The L^2 -Euclidean norm, and the associated scalar product are denoted, respectively, by $\|\cdot\|_2$ and $(\cdot, \cdot)_2$.

2. STATEMENT OF THE ILL-POSED PROBLEM CONCERNING A DEGENERATE PARABOLIC OPERATOR IN $\Omega \times \mathcal{O}$

Let $\Omega \subset \mathbb{R}^M$ and $\mathcal{O} \subset \mathbb{R}^N$ be two open sets of classes C^3 and C^2 , respectively, the first being bounded, the latter unbounded. In particular, also $\mathcal{O} = \mathbb{R}^N$ is allowed.

For any fixed $T > 0$, we consider the following problem: *look for a function $u \in H^1((0, T); L^2_\mathbb{C}(\Omega \times \mathcal{O})) \cap L^2((0, T); \mathcal{H}^2_\mathbb{C}(\Omega \times \mathcal{O}))$ satisfying the following boundary value problem:*

$$\begin{cases} D_t u(t, x, y) = \operatorname{div}_x(a(x) \nabla_x u(t, x, y)) + c(x, y) \cdot \nabla_x u(t, x, y) + b_0(x, y) u(t, x, y) \\ \quad + b(y) \cdot \nabla_y u(t, x, y) + g(t, x, y), & (t, x, y) \in [0, T] \times \Omega \times \mathcal{O} =: Q_T, \\ u(t, x, y) = h(t, x, y), & (t, x, y) \in [0, T] \times \partial_*(\Omega \times \mathcal{O}), \\ D_\nu u(t, x, y) = D_\nu h(t, x, y), & (t, x, y) \in [0, T] \times \Gamma \times \mathcal{O}. \end{cases} \quad (2.1)$$

Here, $\partial_*(\Omega \times \mathcal{O}) = (\partial\Omega \times \mathcal{O}) \cup (\Omega \times \partial\mathcal{O})$, Γ is an open subset of $\partial\Omega$, $\nu = \nu(x)$ denotes the outward normal unit-vector at $x \in \Gamma$ and

$$\mathcal{H}^2_\mathbb{C}(\Omega \times \mathcal{O}) = \{z \in H^2_\mathbb{C}(\Omega \times \mathcal{O}) : |b| |\nabla_y z| \in L^2(\Omega \times \mathcal{O})\}.$$

The hypotheses on the coefficients a , b_0 , b , c and the data g and h are listed here below.

Hypothesis 2.1. *The following conditions are satisfied.*

- (i) $a \in W^{2,\infty}(\Omega)$ and there exists a positive constant a_0 such that $|a(x)| \geq a_0$ for any $x \in \Omega$;
- (ii) $b_0 \in L^\infty_{\mathbb{C}}(\Omega \times \mathcal{O})$;
- (iii) $b = (b_1, \dots, b_N) \in (W^{1,\infty}_{\text{loc}}(\mathcal{O}))^N$ and $\text{div } b \in L^\infty(\mathcal{O})$;
- (iv) $c = (c_1, \dots, c_M) \in (L^\infty_{\mathbb{C}}(\Omega \times \mathcal{O}))^M$;
- (v) $g \in L^2_{\mathbb{C}}(Q_T)$;
- (vi) $h \in H^1((0, T); L^2_{\mathbb{C}}(\Omega \times \mathcal{O})) \cap L^2((0, T); \mathcal{H}^2_{\mathbb{C}}(\Omega \times \mathcal{O}))$.

Performing the translation $v = u - h$, we can change our problem to one with vanishing boundary value data: look for a function $v \in H^1((0, T); L^2_{\mathbb{C}}(\Omega \times \mathcal{O})) \cap L^2((0, T); \mathcal{H}^2_{\mathbb{C}}(\Omega \times \mathcal{O}))$ satisfying the boundary value problem

$$\begin{cases} D_t v(t, x, y) = \text{div}_x(a(x) \nabla_x v(t, x, y)) + c(x, y) \cdot \nabla_x v(t, x, y) \\ \quad + b(y) \cdot \nabla_y v(t, x, y) + b_0(x, y) v(t, x, y) + \tilde{g}(t, x, y), & (t, x, y) \in Q_T, \\ v(t, x, y) = 0, & (t, x, y) \in [0, T] \times \partial_*(\Omega \times \mathcal{O}), \\ D_\nu v(t, x, y) = 0, & (t, x, y) \in [0, T] \times \Gamma \times \mathcal{O}, \end{cases} \quad (2.2)$$

where

$$\begin{aligned} \tilde{g}(t, x, y) = & g(t, x, y) - D_t h(t, x, y) + \text{div}_x(a(x) \nabla_x h(t, x, y)) + c(x, y) \cdot \nabla_x h(t, x, y) \\ & + b(y) \cdot \nabla_y h(t, x, y) + b_0(x, y) h(t, x, y), \end{aligned} \quad (2.3)$$

for any $(t, x, y) \in (0, T) \times \Omega \times \mathcal{O}$.

Remark 2.2. If function a is *strictly positive*, then the differential equation in (2.2) is *forward degenerate parabolic*, while if function a is *strictly negative*, then the differential equation in (2.2) is *backward parabolic*. However, in the present case this is not a trouble at all. Indeed, introducing the new unknown

$$\tilde{v}(t, x, y) = v(-t, x, y), \quad (t, x, y) \in [0, T] \times \Omega \times \mathcal{O},$$

it is immediate to check that \tilde{v} solves the problem (2.2) with $a(x)$, $b(y)$, $c(x, y)$, $b_0(x, y)$, $\tilde{g}(t, x, y)$ being replaced, respectively, by $-a(x)$, $-b(y)$, $-c(x, y)$, $-b_0(x, y)$, $-\tilde{g}(-t, x, y)$, i.e., \tilde{v} solves a problem with a differential *forward degenerate parabolic* equation.

3. CARLEMAN ESTIMATES FOR THE ILL-POSED PROBLEM (2.1)

In view of Remark 2.2, in this section we assume that function a is *strictly positive*, i.e., $a(x) \geq a_0 > 0$ for any $x \in \overline{\Omega}$.

In order to obtain a Carleman estimate related to the domain $\Omega \times \mathcal{O}$ we need a weight function, defined on $\overline{\Omega}$, with special properties. The existence of such a function is proved by extending [22, Lemma 1.1] to the C^3 -case and then [23, Lemma 2.3]. This can be done without a great efforts. For this reasons, the details are left to the reader.

Lemma 3.1. *There exists a function $\psi \in C^3(\overline{\Omega})$ with the following properties:*

- (i) ψ is positive in Ω ;
- (ii) $|\nabla_x \psi(x)| > 0$ for any $x \in \overline{\Omega}$;
- (iii) $D_\nu \psi(x) \leq 0$ for any $x \in \partial\Omega \setminus \Gamma$.

For any $\rho \geq 1$ we set

$$\varphi_\rho(t, x) = \frac{e^{\rho\psi(x)} - e^{2\rho\|\psi\|_\infty}}{t(T-t)}, \quad (t, x) \in (0, T) \times \overline{\Omega}, \quad (3.1)$$

and, for simplicity, in the rest of this section we set $\ell(t) = t(T-t)$.

In the following lemma we list some crucial estimate of the function φ_ρ that we need in the proof of the Carleman estimates.

Lemma 3.2. *For any $x \in \overline{\Omega}$,*

$$\lim_{t \rightarrow 0^+} \varphi_\rho(t, x) = \lim_{t \rightarrow T^-} \varphi_\rho(t, x) = +\infty.$$

Further, let α denote the positive infimum of the function $|\nabla_x \psi|$. Then, the following pointwise inequalities hold true:

$$\ell^{-1} \leq \frac{1}{\alpha\rho} |\nabla_x \varphi_\rho|, \quad (3.2)$$

$$|D_t \varphi_\rho| \leq \frac{T}{\alpha^2} e^{2\rho\|\psi\|_\infty} |\nabla_x \varphi_\rho|^2 \leq \frac{T^3}{4\alpha^3} e^{2\rho\|\psi\|_\infty} |\nabla_x \varphi_\rho|^3, \quad (3.3)$$

$$|D_t^2 \varphi_\rho| \leq \frac{T^2}{2\alpha^3} e^{2\rho\|\psi\|_\infty} |\nabla_x \varphi_\rho|^3, \quad (3.4)$$

$$|D_t \nabla_x \varphi_\rho| \leq \frac{T^3}{4\alpha^2} |\nabla_x \varphi_\rho|^3, \quad (3.5)$$

$$\begin{aligned} |D_{x_i x_j} \varphi_\rho| &\leq \frac{T^2}{4\alpha^2} (\|\psi\|_{2,\infty} + \alpha\|\psi\|_{1,\infty}) |\nabla_x \varphi_\rho|^2 \\ &\leq \frac{T^4}{16\alpha^3} (\|\psi\|_{2,\infty} + \alpha\|\psi\|_{1,\infty}) |\nabla_x \varphi_\rho|^3, \end{aligned} \quad (3.6)$$

$$\begin{aligned} |D_{x_i x_j x_k} \varphi_\rho| &\leq \frac{T^2}{4\alpha^2} (\|\psi\|_{3,\infty} + 3\alpha\|\psi\|_{2,\infty} + \alpha^2\|\psi\|_{1,\infty}) |\nabla_x \varphi_\rho|^2 \\ &\leq \frac{T^4}{16\alpha^3} (\|\psi\|_{3,\infty} + 3\alpha\|\psi\|_{2,\infty} + \alpha^2\|\psi\|_{1,\infty}) |\nabla_x \varphi_\rho|^3, \end{aligned} \quad (3.7)$$

for any $i, j, k = 1, \dots, M$.

Proof. The proof of (3.2) is straightforward. We limit ourselves to proving (3.3), (3.6) and (3.7), the other estimates being completely similar to prove.

Since

$$D_t \varphi_\rho(t, \cdot) = (t - 2T) \frac{e^{2\rho\|\psi\|_\infty} - e^{\rho\psi}}{(\ell(t))^2}, \quad t \in (0, T),$$

we can estimate, using (3.2),

$$|D_t \varphi_\rho(t, \cdot)| \leq \frac{T}{(\ell(t))^2} e^{2\rho\|\psi\|_\infty} \leq \frac{T}{\alpha^2 \rho^2} e^{2\rho\|\psi\|_\infty} |\nabla_x \varphi_\rho|^2,$$

for any $t \in (0, T)$, which gives the first inequality in (3.3) since $\rho \geq 1$.

To prove the second inequality in (3.3) it suffices to use again (3.2) and the estimate $\|\ell\|_\infty \leq T^2/4$ to obtain

$$\frac{T}{\alpha^2} e^{2\rho\|\psi\|_\infty} |\nabla_x \varphi_\rho|^2 = \frac{T}{\alpha^2} e^{2\rho\|\psi\|_\infty} \frac{\ell(t)}{\alpha\rho} \frac{\alpha\rho}{\ell(t)} |\nabla_x \varphi_\rho|^2 \leq \frac{T^3}{4\alpha^3} e^{2\rho\|\psi\|_\infty} |\nabla_x \varphi_\rho|^3.$$

Let us finally prove the first inequalities in (3.6) and (3.7), the other two inequalities in (3.6) and (3.7) then follow from these ones and (3.2). For this purpose we observe that

$$D_{x_i x_j} \varphi_\rho = \frac{\rho}{\ell} (D_{x_i x_j} \psi + \rho D_{x_i} \psi D_{x_j} \psi) e^{\rho\psi}.$$

Hence,

$$\begin{aligned}
|D_{x_i x_j} \varphi_\rho| &\leq \frac{\rho}{\ell} \|\psi\|_{2,\infty} e^{\rho\psi} + \rho \|\psi\|_{1,\infty} |\nabla_x \varphi_\rho| \\
&= \frac{\rho}{\ell} \frac{\ell^2}{\alpha^2 \rho^2} \frac{\alpha^2 \rho^2}{\ell^2} e^{\rho\psi} \|\psi\|_{2,\infty} + \rho \frac{\ell}{\alpha \rho} \frac{\alpha \rho}{\ell} \|\psi\|_{1,\infty} |\nabla_x \varphi_\rho| \\
&\leq \frac{\ell}{\alpha^2 \rho} \left(\frac{\rho^2}{\ell^2} |\nabla_x \psi|^2 e^{2\rho\psi} \right) \|\psi\|_{2,\infty} + \frac{\ell}{\alpha} \frac{\alpha \rho}{\ell} \|\psi\|_{1,\infty} |\nabla_x \varphi_\rho| \\
&\leq \frac{T^2}{4\alpha^2} \|\psi\|_{2,\infty} |\nabla_x \varphi_\rho|^2 + \frac{T^2}{4\alpha} \|\psi\|_{1,\infty} |\nabla_x \varphi_\rho|^2.
\end{aligned}$$

The first estimate in (3.6) follows at once.

Similarly, one has

$$\begin{aligned}
D_{x_i x_j x_k} \varphi_\rho &= \frac{\rho}{\ell} [D_{x_i x_j x_k} \psi + \rho(D_{x_i x_j} \psi D_{x_k} \psi + D_{x_j x_k} \psi D_{x_i} \psi + D_{x_i x_k} \psi D_{x_j} \psi) \\
&\quad + \rho^2 D_{x_i} \psi D_{x_j} \psi D_{x_k} \psi] e^{\rho\psi}.
\end{aligned}$$

Hence, arguing as above, one gets

$$\begin{aligned}
|D_{x_i x_j x_k} \varphi_\rho| &\leq \frac{\rho}{\ell} \|\psi\|_{3,\infty} e^{\rho\psi} + 3\rho \|\psi\|_{2,\infty} |\nabla_x \varphi_\rho| + \rho^2 \|\psi\|_{1,\infty} |\nabla_x \psi| |\nabla_x \varphi_\rho| \\
&\leq \frac{\ell}{\alpha^2 \rho} \left(\frac{\rho^2}{\ell^2} |\nabla_x \psi|^2 e^{2\rho\psi} \right) \|\psi\|_{3,\infty} + 3 \frac{\ell}{\alpha} \frac{\alpha \rho}{\ell} \|\psi\|_{2,\infty} |\nabla_x \varphi_\rho| \\
&\quad + \|\psi\|_{1,\infty} \left(\frac{\rho}{\ell} |\nabla_x \psi| e^{\rho\psi} \right) \rho \ell |\nabla_x \varphi_\rho| \\
&\leq \frac{T^2}{4\alpha^2} \|\psi\|_{3,\infty} |\nabla_x \varphi_\rho|^2 + \frac{3T^2}{4\alpha} \|\psi\|_{2,\infty} |\nabla_x \varphi_\rho|^2 + \frac{T^2}{4} \rho \|\psi\|_{1,\infty} |\nabla_x \varphi_\rho|^2.
\end{aligned}$$

□

The main result of this section is the following theorem.

Theorem 3.3 (Carleman estimates). *There exist two positive constants $\tilde{\lambda}_0 = \tilde{\lambda}_0(a_0, \|a\|_{2,\infty}, \|b_0\|_\infty, \|\operatorname{div} b\|_\infty, \|c\|_\infty, \|\psi\|_{3,\infty}, \alpha, T)$, $\rho_0 = \rho_0(a_0, \|a\|_{1,\infty}, \|\psi\|_{2,\infty}, \alpha)$ for which the following estimate holds for all $\lambda \geq \tilde{\lambda}_0$ and all $v \in H^1((0, T); L^2_\mathbb{C}(\Omega \times \mathcal{O})) \cap L^2((0, T); \mathcal{H}^2_\mathbb{C}(\Omega \times \mathbb{R}^N))$:*

$$\int_{Q_T} (\lambda |\nabla_x \varphi_{\rho_0}| |\nabla_x v|^2 + \lambda^3 |\nabla_x \varphi_{\rho_0}|^3 v^2) e^{2\lambda \varphi_{\rho_0}} dt dx dy \leq 64 \int_{Q_T} |\mathcal{P}v|^2 e^{2\lambda \varphi_{\rho_0}} dt dx dy. \quad (3.8)$$

Here, $\mathcal{P}v = D_t v - \operatorname{div}_x(a \nabla_x v) - b \cdot \nabla_y v - c \cdot \nabla_x v - b_0 v$.

Corollary 3.4. *Suppose that g and h vanish in Q_T . If u is a solution to problem (2.1) in Q_T , then $u = 0$.*

Proof. If $g = h = 0$ in Q_T , then $\tilde{g} = 0$ in Q_T . Estimate (3.8) yields $v = 0$ in Q_T . This, in turn, implies, via the equality $v = u - h$, $u = 0$ in Q_T , so that the principle of *unique continuation* holds for the solution to problem (2.1). □

The proof of Theorem 3.3 is a consequence of the following theorem for real-valued functions and the principal part \mathcal{P}_0 of operator \mathcal{P} .

Theorem 3.5 (Carleman estimates in a simplified case). *Two positive constants $\lambda_0 = \lambda_0(a_0, \|a\|_{2,\infty}, \|\operatorname{div} b\|_\infty, \|\psi\|_{3,\infty}, \alpha, T)$ and $\rho_0 = \rho_0(a_0, \|a\|_{1,\infty}, \|\psi\|_{2,\infty}, \alpha)$ exist such that*

$$\int_{Q_T} (\lambda |\nabla_x \varphi_{\rho_0}| |\nabla_x v|^2 + \lambda^3 |\nabla_x \varphi_{\rho_0}|^3 v^2) e^{2\lambda \varphi_{\rho_0}} dt dx dy \leq \frac{32}{3} \int_{Q_T} |\mathcal{P}_0 v|^2 e^{2\lambda \varphi_{\rho_0}} dt dx dy, \quad (3.9)$$

for all $v \in H^1((0, T); L^2(\Omega \times \mathcal{O})) \cap L^2((0, T); \mathcal{H}^2(\Omega \times \mathcal{O}))$ and all $\lambda \geq \lambda_0$. Here, $\mathcal{P}_0 v = D_t v - \operatorname{div}_x(a \nabla_x v) - b \cdot \nabla_y v$.

Proof of Theorem 3.3. Fix $v \in H^1((0, T); L^2_{\mathbb{C}}(\Omega \times \mathcal{O})) \cap L^2((0, T); \mathcal{H}^2_{\mathbb{C}}(\Omega \times \mathcal{O}))$. Then, $v_1 = \operatorname{Re} v$ and $v_2 = \operatorname{Im} v$ belong to the space $H^1((0, T); L^2(\Omega \times \mathcal{O})) \cap L^2((0, T); \mathcal{H}^2(\Omega \times \mathcal{O}))$. Consequently, both v_1 and v_2 satisfy estimate (3.9), where v is replaced with v_j , $j = 1, 2$. Since the coefficients of the operator \mathcal{P}_0 are all real-valued functions, $|\mathcal{P}_0 v|^2 = |\mathcal{P}_0 v_1|^2 + |\mathcal{P}_0 v_2|^2$. Therefore, summing the Carleman estimates for v_1 and v_2 , we get (3.9) for v .

To show that v satisfies (3.8), we take advantage of the elementary inequalities

$$|\mathcal{P}_0 v|^2 \leq 3|\mathcal{P}v|^2 + 3\|c\|_{\infty}^2 |\nabla_x v|^2 + 3\|b_0\|_{\infty}^2 |v|^2$$

and of (3.2), with $\rho = \rho_0$, implying

$$\begin{aligned} & \int_{Q_T} (\|c\|_{\infty}^2 |\nabla_x v|^2 + \|b_0\|_{\infty}^2 |v|^2) e^{2\lambda\varphi_{\rho_0}} dt dx dy \\ & \leq \frac{T^2}{4\alpha\rho_0} \|c\|_{\infty}^2 \int_{Q_T} |\nabla_x \varphi_{\rho_0}| |\nabla_x v|^2 e^{2\lambda\varphi_{\rho_0}} dt dx dy \\ & \quad + \frac{T^6}{64\alpha^3\rho_0^3} \|b_0\|_{\infty}^2 \int_{Q_T} |\nabla_x \varphi_{\rho_0}|^3 |v|^2 e^{2\lambda\varphi_{\rho_0}} dt dx dy. \end{aligned}$$

Choose now

$$\lambda \geq \frac{16T^2}{\alpha\rho_0} \|c\|_{\infty}^2, \quad \lambda^3 \geq \frac{T^6}{\alpha^3\rho_0^3} \|b_0\|_{\infty}^2.$$

Then, (3.8) holds with $\tilde{\lambda}_0$ being defined by

$$\tilde{\lambda}_0 = \max \left\{ \lambda_0, \frac{16T^2}{\alpha\rho_0} \|c\|_{\infty}^2, \frac{T^2}{\alpha\rho_0} \|b\|_{\infty}^{2/3} \right\}.$$

□

Proof of Theorem 3.5. Let $w_{\rho} : Q_T \rightarrow \mathbb{R}$ be the function defined by

$$w_{\rho}(t, x, y) = v(t, x, y) e^{\lambda\varphi_{\rho}(t, x)}, \quad (t, x, y) \in (0, T) \times \Omega \times \mathcal{O}, \quad (3.10)$$

depending on the positive parameters λ and ρ . According to the definitions of φ_{ρ} we easily deduce that w_{ρ} has the same degree of smoothness as v . Moreover, $\|w_{\rho}(t, \cdot)\|_{H^2(\Omega \times \mathcal{O})}$ and $w_{\rho}(t, x, y)$ tend to 0 as $t \rightarrow 0^+$ and $t \rightarrow T^-$, the latter one for any $(x, y) \in \Omega \times \mathcal{O}$.

For almost all the proof, to avoid cumbersome notation, we simply write w and φ instead of w_{ρ} and φ_{ρ} .

Define the linear operator \mathcal{L}_{λ} by

$$\mathcal{L}_{\lambda} w = e^{\lambda\varphi} \mathcal{P}_0(w e^{-\lambda\varphi}). \quad (3.11)$$

After some computations we can split \mathcal{L}_{λ} into the sum $\mathcal{L}_{\lambda} = \mathcal{L}_{1,\lambda}^+ + (\mathcal{L}_{1,\lambda}^- - b \cdot \nabla_y)$, where

$$\mathcal{L}_{1,\lambda}^+ w = -\operatorname{div}_x(a \nabla_x w) - \lambda(D_t \varphi + \lambda a |\nabla_x \varphi|^2) w =: \sum_{k=1}^2 \mathcal{L}_{1,\lambda,k}^+ w, \quad (3.12)$$

$$\mathcal{L}_{1,\lambda}^- w = D_t w + 2\lambda a \nabla_x \varphi \cdot \nabla_x w + \lambda(a \Delta_x \varphi + \nabla_x a \cdot \nabla_x \varphi) w =: \sum_{k=1}^3 \mathcal{L}_{1,\lambda,k}^- w. \quad (3.13)$$

Clearly,

$$\|\mathcal{L}_{\lambda} w\|_2^2 = \|\mathcal{L}_{1,\lambda}^+ w\|_2^2 + \|\mathcal{L}_{1,\lambda}^- w - b \cdot \nabla_y w\|_2^2 + 2(\mathcal{L}_{1,\lambda}^+ w, \mathcal{L}_{1,\lambda}^- w - b \cdot \nabla_y w)_2$$

$$\geq \|\mathcal{L}_{1,\lambda}^+ w\|_2^2 + 2(\mathcal{L}_{1,\lambda}^+ w, \mathcal{L}_{1,\lambda}^- w)_2 - 2(\mathcal{L}_{1,\lambda}^+ w, b \cdot \nabla_y w)_2. \quad (3.14)$$

To rewrite the terms $(\mathcal{L}_{1,\lambda}^+ w, \mathcal{L}_{1,\lambda}^- w)_2$ and $(\mathcal{L}_{1,\lambda}^+ w, b \cdot \nabla_y w)_2$ in a more convenient way, we perform several integrations by parts.

As the rest of the proof is rather long, we split it into five steps and, for notational convenience, we set

$$\mathcal{J}_1(w) = \int_{Q_T} |\nabla_x \varphi|^3 w^2 dt dx dy, \quad \mathcal{J}_2(w) = \int_{Q_T} |\nabla_x \varphi| |\nabla_x w|^2 dt dx dy. \quad (3.15)$$

Moreover, we denote by C_j positive constants which depend only on the quantities in brackets.

Step 1: the term $2(\mathcal{L}_{1,\lambda}^+ w, \mathcal{L}_{1,\lambda}^- w)_2$. We split this term into the sum of the addenda $(\mathcal{L}_{1,\lambda,i}^+ w, \mathcal{L}_{1,\lambda,j}^- w)_2$ ($i = 1, 2, j = 1, 2, 3$).

We claim that $(\mathcal{L}_{1,\lambda,1}^+ w, \mathcal{L}_{1,\lambda,1}^- w)_2 = 0$. To prove the claim, we need to integrate by parts. For this purpose, we approximate function v by a sequence $\{v_n\} \subset H^1((0, T); L^2(\mathcal{O}; H^2(\Omega) \cap H_0^1(\Omega)))$, converging to v in $L^2(Q_T)$, together with its first-order time derivative and first- and second-order spatial derivatives with respect to x . Set $w_n = e^{\lambda \varphi} v_n$ and observe that, integrating by parts with respect to the variable t and recalling that, for any $n \in \mathbb{N}$, $w_n = 0$ on $(0, T) \times \partial\Omega \times \mathcal{O}$ and $\nabla_x w_n(t, \cdot, \cdot)$ tends to 0 in $(L^2(\Omega \times \mathcal{O}))^N$ as $t \rightarrow 0^+$ and $t \rightarrow T^-$, we easily see that

$$\begin{aligned} \int_{Q_T} \operatorname{div}_x(a \nabla_x w_n) D_t w_n dt dx dy &= - \int_{Q_T} a \nabla_x w \cdot D_t \nabla_x w_n dt dx dy \\ &= - \frac{1}{2} \int_{Q_T} a D_t |\nabla_x w_n|^2 dt dx dy \\ &= 0. \end{aligned}$$

Letting $n \rightarrow +\infty$ gives

$$\int_{Q_T} \operatorname{div}_x(a \nabla_x w) D_t w dt dx dy = 0.$$

As far as the term $2(\mathcal{L}_{1,\lambda,2}^+ w, \mathcal{L}_{1,\lambda,1}^- w)_2$ is concerned, we observe that

$$\begin{aligned} 2(\mathcal{L}_{1,\lambda,2}^+ w, \mathcal{L}_{1,\lambda,1}^- w)_2 &= -2\lambda \int_{Q_T} (D_t \varphi + \lambda a |\nabla_x \varphi|^2) w D_t w dt dx dy \\ &= -\lambda \int_{Q_T} (D_t \varphi + \lambda a |\nabla_x \varphi|^2) D_t (w^2) dt dx dy \\ &= \lambda \int_{Q_T} w^2 (D_t^2 \varphi + \lambda a D_t |\nabla_x \varphi|^2) dt dx dy. \end{aligned}$$

Computing the terms $2(\mathcal{L}_{1,\lambda,1}^+ w, \mathcal{L}_{1,\lambda,2}^- w)_2$ is much trickier. Integrating twice by parts, we can write

$$\begin{aligned} 2(\mathcal{L}_{1,\lambda,1}^+ w, \mathcal{L}_{1,\lambda,2}^- w)_2 &= -4\lambda \int_{(0,T) \times \partial\Omega \times \mathcal{O}} a D_\nu w (a \nabla_x \varphi \cdot \nabla_x w) dt d\sigma(x) dy \\ &\quad + 2\lambda \int_{Q_T} a^2 \nabla_x \varphi \cdot \nabla_x (|\nabla_x w|^2) dt dx dy \\ &\quad + 4\lambda \int_{Q_T} a \sum_{j,k=1}^M D_{x_j} (a D_{x_k} \varphi) (D_{x_k} w D_{x_j} w) dt dx dy \\ &= -4\lambda \int_{(0,T) \times \partial\Omega \times \mathcal{O}} a D_\nu w (a \nabla_x \varphi \cdot \nabla_x w) dt d\sigma(x) dy \\ &\quad + 2\lambda \int_{(0,T) \times \partial\Omega \times \mathcal{O}} a^2 D_\nu \varphi |\nabla_x w|^2 dt d\sigma(x) dy \end{aligned}$$

$$\begin{aligned}
& - 2\lambda \int_{Q_T} \operatorname{div}_x(a^2 \nabla_x \varphi) |\nabla_x w|^2 dt dx dy \\
& + 4\lambda \int_{Q_T} a(\nabla_x a \cdot \nabla_x w)(\nabla_x \varphi \cdot \nabla_x w) dt dx dy \\
& + 4\lambda \int_{Q_T} a^2 \sum_{j,k=1}^M D_{x_j} D_{x_k} \varphi (D_{x_k} w D_{x_j} w) dt dx dy. \quad (3.16)
\end{aligned}$$

To rewrite the integrals on $(0, T) \times \partial\Omega \times \mathcal{O}$, we observe that, since $w \equiv 0$ on $(0, T) \times \partial\Omega \times \mathcal{O}$, $\nabla_x w = D_\nu w \boldsymbol{\nu}$ on $(0, T) \times \partial\Omega \times \mathcal{O}$. Therefore, $\nabla_x \varphi \cdot \nabla_x w = \rho \ell^{-1} e^{\rho\psi} D_\nu \psi D_\nu w$ on the same set. Moreover, $\nabla_x w = 0$ on $(0, T) \times \Gamma \times \mathcal{O}$ and $D_\nu \varphi = \rho \ell^{-1} e^{\rho\psi} D_\nu \psi \leq 0$ on $(0, T) \times (\partial\Omega \setminus \Gamma) \times \mathcal{O}$. We can thus write

$$\begin{aligned}
& - 4\lambda \int_{(0,T) \times \partial\Omega \times \mathcal{O}} a D_\nu w (a \nabla_x \varphi \cdot \nabla_x w) dt d\sigma(x) dy \\
& = 4\rho\lambda \int_{(0,T) \times (\partial\Omega \setminus \Gamma) \times \mathcal{O}} \frac{a^2}{\ell} |D_\nu \psi| (D_\nu w)^2 e^{\rho\psi} dt d\sigma(x) dy \quad (3.17)
\end{aligned}$$

and

$$\begin{aligned}
& 2\lambda \int_{(0,T) \times \partial\Omega \times \mathcal{O}} a^2 D_\nu \varphi |\nabla_x w|^2 dt d\sigma(x) dy \\
& = -2\rho\lambda \int_{(0,T) \times (\partial\Omega \setminus \Gamma) \times \mathcal{O}} \frac{a^2}{\ell} |D_\nu \psi| (D_\nu w)^2 e^{\rho\psi} dt d\sigma(x) dy. \quad (3.18)
\end{aligned}$$

From (3.16), (3.17) and (3.18) we get

$$\begin{aligned}
2(\mathcal{L}_{1,\lambda,1}^+ w, \mathcal{L}_{1,\lambda,2}^- w)_2 & = 2\rho\lambda \int_{(0,T) \times (\partial\Omega \setminus \Gamma) \times \mathcal{O}} \frac{a^2}{\ell} |D_\nu \psi| |D_\nu w|^2 e^{\rho\psi} dt d\sigma(x) dy \\
& - 2\lambda \int_{Q_T} \operatorname{div}_x(a^2 \nabla_x \varphi) |\nabla_x w|^2 dt dx dy \\
& + 4\lambda \int_{Q_T} a(\nabla_x a \cdot \nabla_x w)(\nabla_x \varphi \cdot \nabla_x w) dt dx dy \\
& + 4\lambda \int_{Q_T} a^2 \sum_{j,k=1}^M D_{x_j} D_{x_k} \varphi (D_{x_k} w D_{x_j} w) dt dx dy.
\end{aligned}$$

Further, straightforward integrations by parts, where we take into account that w vanishes on $(0, T) \times \partial\Omega \times \mathcal{O}$, show that

$$\begin{aligned}
2(\mathcal{L}_{1,\lambda,1}^+ w, \mathcal{L}_{1,\lambda,3}^- w)_2 & = 2\lambda \int_{Q_T} a(a \Delta_x \varphi + \nabla_x a \cdot \nabla_x \varphi) |\nabla_x w|^2 dt dx dy \\
& + 2\lambda \int_{Q_T} a w \nabla_x w \cdot \nabla_x (a \Delta_x \varphi + \nabla_x a \cdot \nabla_x \varphi) dt dx dy
\end{aligned}$$

and

$$\begin{aligned}
2(\mathcal{L}_{1,\lambda,2}^+ w, \mathcal{L}_{1,\lambda,2}^- w)_2 & = -2\lambda^2 \int_{Q_T} a(D_t \varphi + \lambda a |\nabla_x \varphi|^2) \nabla_x \varphi \cdot \nabla_x (w^2) dt dx dy \\
& = 2\lambda^2 \int_{Q_T} w^2 \operatorname{div}_x [a(D_t \varphi + \lambda a |\nabla_x \varphi|^2) \nabla_x \varphi] dt dx dy \\
& = 2\lambda^2 \int_{Q_T} w^2 (D_t \varphi + \lambda a |\nabla_x \varphi|^2) \operatorname{div}_x (a \nabla_x \varphi) dt dx dy \\
& + 2\lambda^2 \int_{Q_T} w^2 a \nabla_x \varphi \cdot \nabla_x (D_t \varphi + \lambda a |\nabla_x \varphi|^2) dt dx dy.
\end{aligned}$$

Finally, we have

$$2(\mathcal{L}_{1,\lambda,2}^+ w, \mathcal{L}_{1,\lambda,3}^- w)_2 = -2\lambda^2 \int_{Q_T} w^2 (D_t \varphi + \lambda a |\nabla_x \varphi|^2) \operatorname{div}_x (a \nabla_x \varphi) dt dx dy.$$

Summing the previous formulas we get

$$\begin{aligned} 2(\mathcal{L}_{1,\lambda}^+ w, \mathcal{L}_{1,\lambda}^- w)_2 &= \lambda \int_{Q_T} \mathcal{H}_1(\varphi) w^2 dt dx dy + \lambda^2 \int_{Q_T} \mathcal{H}_2(a, \varphi) w^2 dt dx dy \\ &\quad + \lambda^3 \int_{Q_T} \mathcal{H}_3(a, \varphi) w^2 dt dx dy + 2\lambda \mathcal{K}(a, \varphi, w) \\ &\quad + 2\rho\lambda \int_{(0,T) \times (\partial\Omega \setminus \Gamma) \times \mathcal{O}} \frac{a^2}{\ell} |D_\nu \psi| (D_\nu w)^2 e^{\rho\psi} dt d\sigma(x) dy, \end{aligned} \quad (3.19)$$

where

$$\mathcal{H}_1(\varphi) = D_t^2 \varphi, \quad (3.20)$$

$$\mathcal{H}_2(a, \varphi) = 4a \nabla_x D_t \varphi \cdot \nabla_x \varphi, \quad (3.21)$$

$$\mathcal{H}_3(a, \varphi) = 2a |\nabla_x \varphi|^2 \nabla_x a \cdot \nabla_x \varphi + 2a^2 \nabla_x \varphi \cdot \nabla_x (|\nabla_x \varphi|^2), \quad (3.22)$$

$$\begin{aligned} \mathcal{K}(a, \varphi, w) &= - \int_{Q_T} (a \nabla_x a \cdot \nabla_x \varphi) |\nabla_x w|^2 dt dx dy \\ &\quad + 2 \int_{Q_T} a (\nabla_x a \cdot \nabla_x w) (\nabla_x \varphi \cdot \nabla_x w) dt dx dy \\ &\quad + 2 \int_{Q_T} a^2 \sum_{j,k=1}^M D_{x_j} D_{x_k} \varphi (D_{x_k} w D_{x_j} w) dt dx dy \\ &\quad + \int_{Q_T} a w \nabla_x w \cdot \nabla_x (a \Delta_x \varphi + \nabla_x a \cdot \nabla_x \varphi) dt dx dy \end{aligned} \quad (3.23)$$

Step 2: the term $-2(\mathcal{L}_{1,\lambda}^+ w, b \cdot \nabla_y w)_2$. Since

$$(a \nabla_x w) \cdot \nabla_x (b \cdot \nabla_y w) = \frac{1}{2} a b \cdot \nabla_y |\nabla_x w|^2,$$

recalling that $w = 0$ on $[0, T] \times \partial_*(\Omega \times \mathcal{O})$ implies $\nabla_x w = 0$ on $[0, T] \times \Omega \times \partial\mathcal{O}$ and $\nabla_y w = 0$ on $[0, T] \times \partial\Omega \times \mathcal{O}$, an integration by parts shows that

$$\begin{aligned} -2(\mathcal{L}_{1,\lambda}^+ w, b \cdot \nabla_y w)_2 &= 2 \int_{Q_T} \operatorname{div}_x (a \nabla_x w) (b \cdot \nabla_y w) dt dx dy \\ &\quad + 2\lambda \int_{Q_T} (D_t \varphi + \lambda a |\nabla_x \varphi|^2) w (b \cdot \nabla_y w) dt dx dy \\ &= - \int_{Q_T} a b \cdot \nabla_y (|\nabla_x w|^2) dt dx dy \\ &\quad - \lambda \int_{Q_T} (\operatorname{div} b) (D_t \varphi + \lambda a |\nabla_x \varphi|^2) w^2 dt dx dy \\ &= \int_{Q_T} a (\operatorname{div} b) |\nabla_x w|^2 dt dx dy \\ &\quad - \lambda \int_{Q_T} (\operatorname{div} b) (D_t \varphi + \lambda a |\nabla_x \varphi|^2) w^2 dt dx dy. \end{aligned} \quad (3.24)$$

Summing up, from formulae (3.14), (3.19) and (3.24), we obtain the following estimate from below for the norm of $\mathcal{L}_\lambda w$:

$$\|\mathcal{L}_\lambda w\|_2^2 \geq \mathcal{J}_1(w) + \|\mathcal{L}_{1,\lambda}^+ w\|_2^2, \quad (3.25)$$

where

$$\begin{aligned} \mathcal{J}_1(w) = & \overline{\mathcal{K}}(a, \varphi, w, \lambda) + \lambda \int_{Q_T} \overline{\mathcal{H}}_1(b, \varphi) w^2 dt dx dy \\ & + \lambda^2 \int_{Q_T} \overline{\mathcal{H}}_2(a, b, \varphi) w^2 dt dx dy + \lambda^3 \int_{Q_T} \overline{\mathcal{H}}_3(a, \varphi) w^2 dt dx dy \end{aligned}$$

and

$$\overline{\mathcal{H}}_1(b, \varphi) = \mathcal{H}_1(\varphi) - (\operatorname{div} b) D_t \varphi, \quad (3.26)$$

$$\overline{\mathcal{H}}_2(a, b, \varphi) = \mathcal{H}_2(a, \varphi) - a(\operatorname{div} b) |\nabla_x \varphi|^2, \quad (3.27)$$

$$\overline{\mathcal{H}}_3(a, \varphi) = \mathcal{H}_3(a, \varphi), \quad (3.28)$$

$$\overline{\mathcal{K}}(a, b, \varphi, w, \lambda) = 2\lambda \mathcal{K}(a, \varphi, w) + \int_{Q_T} a(\operatorname{div} b) |\nabla_x w|^2 dt dx dy. \quad (3.29)$$

Step 3: estimate of $\mathcal{J}_1(w)$. As a first step, taking advantage of the formula

$$D_{x_k} D_{x_j} \varphi = \frac{\rho}{\ell} e^{\rho\psi} (D_{x_k} D_{x_j} \psi + \rho D_{x_j} \psi D_{x_k} \psi),$$

we obtain (cf. (3.1))

$$\begin{aligned} \sum_{j,k=1}^M D_{x_j} D_{x_k} \varphi (D_{x_k} w D_{x_j} w) &= \frac{\rho}{\ell} e^{\rho\psi} \sum_{j,k=1}^M D_{x_j} D_{x_k} \psi (D_{x_k} w D_{x_j} w) \\ &\quad + \frac{\rho^2}{\ell} e^{\rho\psi} (\nabla \psi \cdot \nabla w)^2 \\ &= |\nabla_x \psi|^{-1} |\nabla_x \varphi| \sum_{j,k=1}^M D_{x_j} D_{x_k} \psi (D_{x_k} w D_{x_j} w) \\ &\quad + \frac{\rho^2}{\ell} e^{\rho\psi} (\nabla \psi \cdot \nabla w)^2 \\ &\geq -\frac{M}{\alpha} \|\psi\|_{2,\infty} |\nabla_x \varphi| |\nabla w|^2, \end{aligned}$$

where α is the infimum of the function $|\nabla \psi|$ over Ω . Since

$$\begin{aligned} |a \nabla_x (a \Delta_x \varphi + \nabla_x a \cdot \nabla_x \varphi)| &\leq \|a\|_\infty \left(\sqrt{M} \|a\|_{1,\infty} |\Delta_x \varphi| + \|a\|_\infty |\nabla_x \Delta_x \varphi| \right. \\ &\quad \left. + M \|a\|_{2,\infty} |\nabla_x \varphi| + M^{3/2} \|a\|_{1,\infty} \sum_{i,j=1}^M |D_{ij} \varphi| \right) \\ &\leq C_1 (\|a\|_{2,\infty}, \|\psi\|_{3,\infty}, \rho, \alpha, T) |\nabla_x \varphi|^2, \end{aligned}$$

where we used (3.2) and the first inequalities in (3.6) and (3.7), we can estimate (using Hölder inequality)

$$\begin{aligned} & 2\lambda \int_{Q_T} a w \nabla_x w \cdot \nabla_x (a \Delta_x \varphi + \nabla_x a \cdot \nabla_x \varphi) dt dx dy \\ & \geq -2\lambda \int_{Q_T} |w| |\nabla_x w| |a \nabla_x (a \Delta_x \varphi + \nabla_x a \cdot \nabla_x \varphi)| dt dx dy \\ & \geq -2C_1 (\|a\|_{2,\infty}, \|\psi\|_{3,\infty}, \rho, \alpha, T) \int_{Q_T} (\lambda^{3/4} |\nabla_x \varphi|^{3/2} |w|) (\lambda^{1/4} |\nabla_x \varphi|^{1/2} |\nabla_x w|) dt dx dy \end{aligned}$$

$$\geq -C_1(\|a\|_{2,\infty}, \|\psi\|_{3,\infty}, \rho, \alpha, T)\lambda^{3/2}\mathcal{J}_1(w) - C_1(\|a\|_{2,\infty}, \|\psi\|_{3,\infty}, \rho, \alpha, T)\lambda^{1/2}\mathcal{J}_2(w),$$

where $\mathcal{J}_1(w)$ and $\mathcal{J}_2(w)$ are defined in (3.15). We conclude that

$$2\lambda\mathcal{K}(a, \varphi, w) \geq -C_1(\|a\|_{2,\infty}, \|\psi\|_{3,\infty}, \rho, \alpha, T)\lambda^{3/2}\mathcal{J}_1(w) \\ - \left[C_2(\|a\|_{1,\infty}, \|\psi\|_{2,\infty}, \alpha)\lambda + C_1(\|a\|_{2,\infty}, \|\psi\|_{3,\infty}, \rho, \alpha, T)\lambda^{1/2} \right] \mathcal{J}_2(w).$$

Moreover, from (3.2), and recalling that $\rho \geq 1$, we get

$$\left| \int_{Q_T} a(\operatorname{div} b) |\nabla_x w|^2 dt dx dy \right| \leq \|a\|_\infty \|\operatorname{div} b\|_\infty \int_{Q_T} |\nabla_x w|^2 dt dx dy \\ \leq \frac{T^2}{4\alpha} \|a\|_\infty \|\operatorname{div} b\|_\infty \mathcal{J}_2(w).$$

Therefore, it follows that

$$\overline{\mathcal{K}}(a, b, \varphi, \nabla_x w, \lambda) \geq -C_1(\|a\|_{2,\infty}, \|\psi\|_{3,\infty}, \rho, \alpha, T)\lambda^{3/2}\mathcal{J}_1(w) \\ - \left[C_2(\|a\|_{1,\infty}, \|\psi\|_{2,\infty}, \alpha)\lambda + C_1(\|a\|_{2,\infty}, \|\psi\|_{3,\infty}, \rho, \alpha, T)\lambda^{1/2} \right. \\ \left. + \frac{T^2}{4\alpha} \|a\|_\infty \|\operatorname{div} b\|_\infty \right] \mathcal{J}_2(w). \quad (3.30)$$

We now consider the terms containing $\overline{\mathcal{H}}_1(b, \varphi)$, $\overline{\mathcal{H}}_2(a, b, \varphi)$ and $\overline{\mathcal{H}}_3(a, \varphi)$ (cf. (3.26)-(3.28)). From (3.2), the second inequality in (3.3), (3.4), (3.5) and definitions (3.20)-(3.23)), we deduce the pointwise inequalities

$$|\overline{\mathcal{H}}_1(b, \varphi)| \leq C_3(\|\operatorname{div} b\|_\infty, \rho, \alpha, T) |\nabla_x \varphi|^3, \quad (3.31)$$

$$|\overline{\mathcal{H}}_2(a, b, \varphi)| \leq C_4(\|a\|_\infty, \|\operatorname{div} b\|_\infty, \alpha, T) |\nabla_x \varphi|^3, \quad (3.32)$$

where we have used the condition $\rho \geq 1$, and (recalling that $|\nabla_x \psi| \geq \alpha$)

$$\overline{\mathcal{H}}_3(a, \varphi) = 2a |\nabla_x \varphi|^2 \nabla_x a \cdot \nabla_x \varphi + 2a^2 \nabla_x \varphi \cdot \nabla_x (|\nabla_x \varphi|^2) \\ = 2a |\nabla_x \varphi|^2 (\nabla_x a \cdot \nabla_x \varphi) + 2a^2 \rho^3 \ell^{-3} e^{3\rho\psi} \nabla_x \psi \cdot \nabla_x (|\nabla_x \psi|^2) \\ + 4a^2 \rho^4 \ell^{-3} e^{3\rho\psi} |\nabla_x \psi|^4 \\ \geq 4a_0^2 \alpha \rho |\nabla_x \varphi|^3 - C_5(\|a\|_{1,\infty}, \|\psi\|_{2,\infty}, \alpha) |\nabla_x \varphi|^3. \quad (3.33)$$

Summing up, from (3.30)-(3.33), we get the following estimate from below for $\mathcal{J}_1(w)$:

$$\mathcal{J}_1(w) \geq \left\{ [4a_0^2 \alpha \rho - C_5(\|a\|_{1,\infty}, \|\psi\|_{2,\infty}, \alpha)] \lambda^3 \right. \\ - C_4(\|a\|_\infty, \|\operatorname{div} b\|_\infty, \alpha, T) \lambda^2 - C_1(\|a\|_{2,\infty}, \|\psi\|_{3,\infty}, \rho, \alpha, T) \lambda^{3/2} \\ - C_3(\|\operatorname{div} b\|_\infty, \rho, \alpha, T) \lambda \left. \right\} \mathcal{J}_1(w) \\ - \left(C_2(\|a\|_{1,\infty}, \|\psi\|_{2,\infty}, \alpha) \lambda + C_1(\|a\|_{2,\infty}, \|\psi\|_{3,\infty}, \rho, \alpha, T) \lambda^{1/2} \right. \\ \left. + \frac{T^2}{4\alpha} \|a\|_\infty \|\operatorname{div} b\|_\infty \right) \mathcal{J}_2(w). \quad (3.34)$$

Step 4: estimate of $\|\mathcal{L}_{1,\lambda}^+ w\|_2^2$. Using the inequalities

$$(\operatorname{div}_x(a \nabla_x w))^2 = [\mathcal{L}_{1,\lambda}^+ w + \lambda(D_t \varphi + \lambda a |\nabla_x \varphi|^2) w]^2 \\ \leq 3(\mathcal{L}_{1,\lambda}^+ w)^2 + 3\lambda^2 (D_t \varphi)^2 w^2 + 3\lambda^4 a^2 |\nabla_x \varphi|^4 w^2,$$

(cf. (3.12)), $|\nabla_x \varphi| \geq 4\rho\alpha T^{-2}$ (which follows from (3.2) and the first inequality in (3.3)), we can infer that

$$\frac{T^2}{\rho\alpha\lambda} \|\mathcal{L}_{1,\lambda}^+ w\|_2^2 \geq \int_{Q_T} \lambda^{-1} |\nabla_x \varphi|^{-1} (\operatorname{div}_x(a \nabla_x w))^2 dt dx dy$$

$$- [4\|a\|_\infty^2 \lambda^3 + C_6(\|\psi\|_\infty, \rho, \alpha, T)\lambda] \mathcal{J}_1(w). \quad (3.35)$$

Now we want to show that the integral term in (3.35) can be estimated from below by a positive constant times $\mathcal{J}_2(w)$ minus some terms which can be controlled by means of $\mathcal{J}_2(w)$ and the good term in $\mathcal{J}_2(w)$. For this purpose, we begin by observing that an integration by parts yields

$$\begin{aligned} \rho^{1/2} \lambda \int_{Q_T} a |\nabla_x \varphi| |\nabla_x w|^2 dt dx dy &= \rho^{1/2} \lambda \int_{Q_T} |\nabla_x \varphi| (a \nabla_x w) \cdot \nabla_x w dt dx dy \\ &= -\rho^{1/2} \lambda \int_{Q_T} |\nabla_x \varphi| w \operatorname{div}_x (a \nabla_x w) dt dx dy \\ &\quad - \frac{1}{2} \rho^{1/2} \lambda \int_{Q_T} a \nabla_x (|\nabla_x \varphi|) \cdot \nabla_x (w^2) dt dx dy \\ &= -\rho^{1/2} \lambda \int_{Q_T} |\nabla_x \varphi| w \operatorname{div}_x (a \nabla_x w) dt dx dy \\ &\quad + \frac{1}{2} \rho^{1/2} \lambda \int_{Q_T} w^2 \operatorname{div}_x [a \nabla_x (|\nabla_x \varphi|)] dt dx dy. \end{aligned}$$

Since

$$\begin{aligned} \rho^{1/2} \lambda |\nabla_x \varphi| |w \operatorname{div}_x (a \nabla_x w)| &= (\lambda |\nabla_x \varphi|)^{-1/2} |\operatorname{div}_x (a \nabla_x w)| \rho^{1/2} (\lambda |\nabla_x \varphi|)^{3/2} |w| \\ &\leq \frac{1}{4\varepsilon} \lambda^{-1} |\nabla_x \varphi|^{-1} [\operatorname{div}_x (a \nabla_x w)]^2 + \varepsilon \rho \lambda^3 |\nabla_x \varphi|^3 w^2, \end{aligned}$$

for any $\varepsilon > 0$, we get

$$\begin{aligned} &\rho^{1/2} \lambda \int_{Q_T} a |\nabla_x \varphi| |\nabla_x w|^2 dt dx dy \\ &\leq \frac{1}{4\varepsilon} \int_{Q_T} \lambda^{-1} |\nabla_x \varphi|^{-1} [\operatorname{div}_x (a \nabla_x w)]^2 dt dx dy + \rho \varepsilon \lambda^3 \mathcal{J}_1(w) \\ &\quad + \frac{1}{2} \rho^{1/2} \lambda \int_{Q_T} w^2 \operatorname{div}_x [a \nabla_x (|\nabla_x \varphi|)] dt dx dy \end{aligned}$$

or, equivalently,

$$\begin{aligned} &\int_{Q_T} \lambda^{-1} |\nabla_x \varphi|^{-1} [\operatorname{div}_x (a \nabla_x w)]^2 dt dx dy \\ &\geq 4\varepsilon a_0 \rho^{1/2} \lambda \mathcal{J}_2(w) - 4\varepsilon^2 \rho \lambda^3 \mathcal{J}_1(w) - 2\varepsilon \rho^{1/2} \lambda \int_{Q_T} w^2 \operatorname{div}_x [a \nabla_x (|\nabla_x \varphi|)] dt dx dy. \end{aligned} \quad (3.36)$$

Using the second estimates in (3.6) and (3.7) we can estimate

$$|\operatorname{div}_x [a \nabla_x (|\nabla_x \varphi|)]| \leq C_7(\|a\|_{1,\infty}, \|\psi\|_{3,\infty}, \alpha, T) |\nabla_x \varphi|^3. \quad (3.37)$$

Replacing (3.36) and (3.37) into (3.35) and assuming that

$$\frac{\alpha \lambda}{T^2} \geq 1, \quad (3.38)$$

implying $T^2/(\rho \alpha \lambda) \leq 1$ since $\rho \geq 1$, we get

$$\begin{aligned} \|\mathcal{L}_{1,\lambda}^+ w\|_2^2 &\geq 4\varepsilon a_0 \rho^{1/2} \lambda \mathcal{J}_2(w) - [4\varepsilon^2 \rho \lambda^3 + 4\lambda^3 \|a\|_\infty^2 + \varepsilon C_8(\|a\|_{1,\infty}, \|\psi\|_{3,\infty}, \alpha, T)\lambda \\ &\quad + C_6(\|\psi\|_\infty, \rho, \alpha, T)\lambda] \mathcal{J}_1(w). \end{aligned} \quad (3.39)$$

Step 5: the final step. Under condition (3.38), from formula (3.25) and estimates (3.34) and (3.39) we obtain

$$\|\mathcal{L}_\lambda w\|_2^2 \geq \left\{ [4(a_0^2 \alpha - \varepsilon^2) \rho - C_5(\|a\|_{1,\infty}, \|\psi\|_{2,\infty}, \alpha) - 4\|a\|_\infty^2] \lambda^3 \right.$$

$$\begin{aligned}
& -C_4(\|a\|_\infty, \|\operatorname{div} b\|_\infty, \alpha, T)\lambda^2 - C_1(\|a\|_{2,\infty}, \|\psi\|_{3,\infty}, \rho, \alpha, T)\lambda^{3/2} \\
& -C_3(\|\operatorname{div} b\|_\infty, \rho, \alpha, T)\lambda - \varepsilon C_8(\|a\|_{1,\infty}, \|\psi\|_{3,\infty}, \alpha, T)\lambda \\
& -C_6(\|\psi\|_\infty, \rho, \alpha, T)\lambda \Big\} \mathcal{J}_1(w) \\
& + \left\{ 2 \left[2\varepsilon a_0 \rho^{1/2} - C_2(\|a\|_{1,\infty}, \|\psi\|_{2,\infty}, \alpha) \right] \lambda \right. \\
& \quad \left. - C_1(\|a\|_{2,\infty}, \|\psi\|_{3,\infty}, \rho, \alpha, T)\lambda^{1/2} - \frac{T^2}{4\alpha} \|a\|_\infty \|\operatorname{div} b\|_\infty \right\} \mathcal{J}_2(w).
\end{aligned}$$

First we fix $\varepsilon^2 = a_0^2 \alpha / 2$ and get

$$\begin{aligned}
\|\mathcal{L}_\lambda w\|_2^2 & \geq \left\{ \left[2a_0^2 \alpha \rho - C_5(\|a\|_{1,\infty}, \|\psi\|_{2,\infty}, \alpha) - 4\|a\|_\infty^2 \right] \lambda^3 \right. \\
& - C_4(\|a\|_\infty, \|\operatorname{div} b\|_\infty, \alpha, T)\lambda^2 - C_1(\|a\|_{2,\infty}, \|\psi\|_{3,\infty}, \rho, \alpha, T)\lambda^{3/2} \\
& - C_3(\|\operatorname{div} b\|_\infty, \rho, \alpha, T)\lambda - \frac{a_0 \sqrt{\alpha}}{\sqrt{2}} C_8(\|a\|_{1,\infty}, \|\psi\|_{3,\infty}, \alpha, T)\lambda \\
& - C_6(\|\psi\|_\infty, \rho, \alpha, T)\lambda \Big\} \mathcal{J}_1(w) \\
& + \left\{ 2 \left[\sqrt{2\alpha} a_0^2 \rho^{1/2} - C_2(\|a\|_{1,\infty}, \|\psi\|_{2,\infty}, \alpha) \right] \lambda \right. \\
& \quad \left. - C_1(\|a\|_{2,\infty}, \|\psi\|_{3,\infty}, \rho, \alpha, T)\lambda^{1/2} - \frac{T^2}{4\alpha} \|a\|_\infty \|\operatorname{div} b\|_\infty \right\} \mathcal{J}_2(w).
\end{aligned}$$

We now choose $\rho = \rho_0$ so as to satisfy the inequalities

$$\begin{cases} \rho \geq 1, \\ 2a_0^2 \alpha \rho - C_5(\|a\|_{1,\infty}, \|\psi\|_{2,\infty}, \alpha) - 4\|a\|_\infty^2 \geq 1, \\ \sqrt{2\alpha} a_0^2 \rho^{1/2} - C_2(\|a\|_{1,\infty}, \|\psi\|_{2,\infty}, \alpha) \geq 1. \end{cases}$$

Corresponding to ρ_0 we determine λ_0 such that the following inequalities are satisfied for all $\lambda \geq \lambda_0$:

$$\begin{cases} \lambda^3 - C_4(\|a\|_\infty, \|\operatorname{div} b\|_\infty, \alpha, T)\lambda^2 - C_1(\|a\|_{2,\infty}, \|\psi\|_{3,\infty}, \rho_0, \alpha, T)\lambda^{3/2} \\ - C_3(\|\operatorname{div} b\|_\infty, \rho_0, \alpha, T)\lambda - \frac{a_0 \sqrt{\alpha}}{\sqrt{2}} C_8(\|a\|_{1,\infty}, \|\psi\|_{3,\infty}, \alpha, T)\lambda \\ - C_6(\|\psi\|_\infty, \rho_0, \alpha, T)\lambda \geq \frac{1}{2}\lambda^3, \\ 2\lambda - C_1(\|a\|_{2,\infty}, \|\psi\|_{3,\infty}, \rho_0, \alpha, T)\lambda^{1/2} - \frac{T^2}{4\alpha} \|a\|_\infty \|\operatorname{div} b\|_\infty \geq \frac{1}{8}\lambda, \\ \lambda \geq \frac{T^2}{\alpha}. \end{cases}$$

Consequently, for all $\lambda \geq \lambda_0$ we deduce the estimate

$$\int_{Q_T} \left(\frac{1}{4} \lambda |\nabla_x \varphi_{\rho_0}| |\nabla_x w_{\rho_0}|^2 + \lambda^3 |\nabla_x \varphi_{\rho_0}|^3 w_{\rho_0}^2 \right) dt dx dy \leq 2 \|\mathcal{L}_\lambda w_{\rho_0}\|_2^2,$$

where, from now on, we write the dependence of φ and w on ρ_0 .

We can now come back to our original solution v using formula (3.10). Observe that

$$\begin{aligned}
& \int_{Q_T} |\nabla_x \varphi_{\rho_0}| |\nabla_x w_{\rho_0}|^2 dt dx dy \\
& \geq \int_{Q_T} |\nabla_x \varphi_{\rho_0}| |\nabla_x v|^2 e^{2\lambda \varphi_{\rho_0}} dt dx dy + \lambda^2 \int_{Q_T} |\nabla_x \varphi_{\rho_0}|^3 v^2 e^{2\lambda \varphi_{\rho_0}} dt dx dy
\end{aligned}$$

$$\begin{aligned}
& - \int_{Q_T} (|\nabla_x \varphi_{\rho_0}|^{1/2} |\nabla_x v| e^{\lambda \varphi_{\rho_0}}) (2\lambda |\nabla_x \varphi_{\rho_0}|^{3/2} |v| e^{\lambda \varphi_{\rho_0}}) dt dx dy \\
& \geq \frac{3}{4} \int_{Q_T} |\nabla_x \varphi_{\rho_0}| |\nabla_x v|^2 e^{2\lambda \varphi_{\rho_0}} dt dx dy - 3\lambda^2 \int_{Q_T} |\nabla_x \varphi_{\rho_0}|^3 v^2 e^{2\lambda \varphi_{\rho_0}} dt dx dy,
\end{aligned}$$

where we used the inequality $|\gamma\delta| \leq \gamma^2/4 + \delta^2$ which holds for any $\gamma, \delta \in \mathbb{R}$. Consequently, owing to (3.11), we get

$$\begin{aligned}
& \int_{Q_T} \left(\frac{3}{16} \lambda |\nabla_x \varphi_{\rho_0}| |\nabla_x v|^2 + \frac{1}{4} \lambda^3 |\nabla_x \varphi_{\rho_0}|^3 v^2 \right) e^{2\lambda \varphi_{\rho_0}} dt dx dy \\
& \leq 2 \int_{Q_T} |\mathcal{P}_0 v|^2 e^{2\lambda \varphi_{\rho_0}} dt dx dy.
\end{aligned}$$

The Carleman estimate (3.8) now follows at once. \square

4. A CONTINUOUS DEPENDENCE RESULT FOR THE ILL-POSED PROBLEM (2.1)

Introduce now the family of functions $\sigma_\varepsilon \in W^{1,\infty}((0, T))$, $\varepsilon \in (0, 1/2)$, defined by

$$\sigma_\varepsilon(t) = \begin{cases} 0, & t \in [0, \varepsilon T], \\ \frac{t - \varepsilon T}{\varepsilon T}, & t \in (\varepsilon T, 2\varepsilon T), \\ 1, & t \in [2\varepsilon T, T]. \end{cases} \quad (4.1)$$

Introduce also the function $v_\varepsilon = \sigma_\varepsilon v$, where v is the solution to problem (2.2). It is a simple task to show that $v_\varepsilon \in H^1((0, T); L^2_\mathbb{C}(\Omega \times \mathcal{O})) \cap L^2((0, T); \mathcal{H}^2_\mathbb{C}(\Omega \times \mathcal{O}))$ solves the following initial and boundary-value problem:

$$\begin{cases} D_t v_\varepsilon(t, x, y) = \operatorname{div}_x(a(x) \nabla_x v_\varepsilon(t, x, y)) + c(x, y) \cdot \nabla_x v_\varepsilon(t, x, y) + \sigma'_\varepsilon(t) v(x, y) \\ \quad + b(y) \cdot \nabla_y v_\varepsilon(t, x, y) + b_0(x, y) v_\varepsilon(t, x, y) + \tilde{g}_\varepsilon(t, x, y), & (t, x, y) \in Q_T, \\ v_\varepsilon(t, x, y) = 0, & (t, x, y) \in [0, T] \times \partial_*(\Omega \times \mathcal{O}), \\ D_\nu v_\varepsilon(t, x, y) = 0, & (t, x, y) \in [0, T] \times \Gamma \times \mathcal{O}, \\ v_\varepsilon(0, x, y) = 0, & (x, y) \in \Omega \times \mathcal{O}, \end{cases}$$

where $\tilde{g}_\varepsilon = \sigma_\varepsilon \tilde{g}$. Multiplying the differential equation by $2\overline{v_\varepsilon}$ and integrating once by parts over $\Omega \times \mathcal{O}$ we obtain the identity

$$\begin{aligned}
& \int_{\Omega \times \mathcal{O}} D_t v_\varepsilon(t, x, y) \overline{v_\varepsilon(t, x, y)} dx dy + \int_{\Omega \times \mathcal{O}} a(x) |\nabla_x v_\varepsilon(t, x, y)|^2 dx dy \\
& = \int_{\Omega \times \mathcal{O}} \overline{v_\varepsilon(t, x, y)} (c(x, y) \cdot \nabla_x v_\varepsilon(t, x, y)) dx dy \\
& \quad + \int_{\Omega \times \mathcal{O}} \overline{v_\varepsilon(t, x, y)} (b(y) \cdot \nabla_y v_\varepsilon(t, x, y)) dx dy \\
& \quad + \int_{\Omega \times \mathcal{O}} b_0(x) |v_\varepsilon(t, x, y)|^2 dx dy + \int_{\Omega \times \mathcal{O}} \tilde{g}_\varepsilon(t, x, y) \overline{v_\varepsilon(t, x, y)} dx dy \\
& \quad + \int_{\Omega \times \mathcal{O}} \sigma'_\varepsilon(t) v(t, x, y) \overline{v_\varepsilon(t, x, y)} dx dy,
\end{aligned}$$

for any $t \in (0, T)$. Taking the real part of both the sides of the previous equality and observing that

$$\operatorname{Re} \left(\int_{\Omega \times \mathcal{O}} D_t v_\varepsilon(t, x, y) \overline{v_\varepsilon(t, x, y)} dx dy \right) = \frac{1}{2} D_t \int_{\Omega \times \mathcal{O}} |v_\varepsilon(t, x, y)|^2 dx dy$$

and

$$\begin{aligned}
& \operatorname{Re} \left(\int_{\Omega \times \mathcal{O}} \overline{v_\varepsilon(t, x, y)} (b(y) \cdot \nabla_y v_\varepsilon(t, x, y)) dx dy \right) \\
&= \int_{\Omega \times \mathcal{O}} b(y) \cdot \operatorname{Re}(\overline{v_\varepsilon(t, x, y)} \nabla_y v_\varepsilon(t, x, y)) dx dy \\
&= \frac{1}{2} \int_{\Omega \times \mathcal{O}} b(y) \cdot \nabla_y |v_\varepsilon(t, x, y)|^2 dx dy \\
&= -\frac{1}{2} \int_{\Omega \times \mathcal{O}} (\operatorname{div} b(y)) |v_\varepsilon(t, x, y)|^2 dx dy,
\end{aligned}$$

we get

$$\begin{aligned}
& D_t \|v_\varepsilon(t, \cdot, \cdot)\|_{L^2(\Omega \times \mathcal{O})}^2 + 2 \int_{\Omega \times \mathcal{O}} a(x) |\nabla_x v_\varepsilon(t, x, y)|^2 dx dy \\
&= 2 \int_{\Omega \times \mathcal{O}} \overline{v_\varepsilon(t, x, y)} (c(x, y) \cdot \nabla_x v_\varepsilon(t, x, y)) dx dy \\
&\quad - \int_{\Omega \times \mathcal{O}} (\operatorname{div} b(y)) |v_\varepsilon(t, x, y)|^2 dx dy \\
&\quad + 2 \int_{\Omega \times \mathcal{O}} b_0(x) |v_\varepsilon(t, x, y)|^2 dx dy + 2 \int_{\Omega \times \mathcal{O}} \tilde{g}_\varepsilon(t, x, y) \overline{v_\varepsilon(t, x, y)} dx dy \\
&\quad + 2 \int_{\Omega \times \mathcal{O}} \sigma'_\varepsilon(t) v(t, x, y) \overline{v_\varepsilon(t, x, y)} dx dy,
\end{aligned}$$

for any $t \in (0, T)$. Therefore, using the elementary inequality

$$2|\overline{v_\varepsilon}(c \cdot \nabla_x v_\varepsilon)| \leq 2\|c\|_\infty |v_\varepsilon| |\nabla_x v_\varepsilon| \leq a_0^{-1} \|c\|_\infty^2 |v_\varepsilon|^2 + a_0 |\nabla_x v_\varepsilon|^2,$$

a_0 being the positive constant in Hypothesis 2.1(i), we can estimate

$$\begin{aligned}
& D_t \|v_\varepsilon(t, \cdot, \cdot)\|_{L^2_\varepsilon(\Omega \times \mathcal{O})}^2 + a_0 \|\nabla_x v_\varepsilon(t, \cdot, \cdot)\|_{L^2_\varepsilon(\Omega \times \mathcal{O})}^2 \\
&\leq (\|\operatorname{div} b\|_\infty + a_0^{-1} \|c\|_\infty^2 + 2\|b_0\|_\infty) \|v_\varepsilon(t, \cdot, \cdot)\|_{L^2_\varepsilon(\Omega \times \mathcal{O})}^2 \\
&\quad + 2\|\tilde{g}_\varepsilon(t, \cdot, \cdot)\|_{L^2_\varepsilon(\Omega \times \mathcal{O})} \|v_\varepsilon(t, \cdot, \cdot)\|_{L^2_\varepsilon(\Omega \times \mathcal{O})} + 2|\sigma'_\varepsilon(t)| \|v(t, \cdot, \cdot)\|_{L^2_\varepsilon(\Omega \times \mathcal{O})}^2. \quad (4.2)
\end{aligned}$$

We now fix $\tau \in (0, T]$ and integrate (4.2) with respect to t over $(0, \tau)$. Taking (4.1) into account, we obtain

$$\begin{aligned}
z_\varepsilon(\tau) &:= \|v_\varepsilon(\tau, \cdot, \cdot)\|_{L^2_\varepsilon(\Omega \times \mathcal{O})}^2 + a_0 \|\nabla_x v_\varepsilon\|_{L^2_\varepsilon(Q_\tau)}^2 \\
&\leq (\|\operatorname{div} b\|_\infty + a_0^{-1} \|c\|_\infty^2 + 2\|b_0\|_\infty) \int_0^\tau \|v_\varepsilon(t, \cdot, \cdot)\|_{L^2_\varepsilon(\Omega \times \mathcal{O})}^2 dt \\
&\quad + 2 \int_0^\tau \|\tilde{g}_\varepsilon(t, \cdot, \cdot)\|_{L^2_\varepsilon(\Omega \times \mathcal{O})} \|v_\varepsilon(t, \cdot, \cdot)\|_{L^2_\varepsilon(\Omega \times \mathcal{O})} dt \\
&\quad + \frac{2}{\varepsilon T} \int_{\varepsilon T}^{2\varepsilon T} \|v(t, \cdot, \cdot)\|_{L^2_\varepsilon(\Omega \times \mathcal{O})}^2 dt. \quad (4.3)
\end{aligned}$$

The Carleman estimate (3.8) yields the inequality

$$\int_{\varepsilon T}^{2\varepsilon T} \|v(t, \cdot, \cdot)\|_{L^2_\varepsilon(\Omega \times \mathcal{O})}^2 dt \leq M_1(\varepsilon, T) \|\tilde{g}\|_{L^2_\varepsilon(Q_T)}^2, \quad (4.4)$$

where the constant $M_1(\varepsilon, T)$ depends also on a_0 , $\|a\|_{2,\infty}$, $\|b_0\|_\infty$, $\|\operatorname{div} b\|_\infty$, $\|c\|_\infty$, $\|\psi\|_{3,\infty}$ and α .

From (4.3) and (4.4) we obtain the following integral inequality for function z_ε :

$$z_\varepsilon(\tau) \leq \beta \int_0^\tau z_\varepsilon(t) dt + 2 \int_0^\tau \|\tilde{g}(t, \cdot, \cdot)\|_{L^2_\varepsilon(\Omega \times \mathcal{O})} z_\varepsilon(t)^{1/2} dt + M_2(\varepsilon, T) \|\tilde{g}\|_{L^2_\varepsilon(Q_T)}^2, \quad (4.5)$$

for any $\tau \in (0, T]$, where $\beta = \|\operatorname{div} b\|_\infty + 2a_0^{-1}\|c\|_\infty^2 + 2\|b_0\|_\infty$. Then, we need [2, Theorem 4.9], with $p = 1/2$, which we report here as a lemma.

Lemma 4.1. *Let $z : [0, T] \rightarrow \mathbb{R}$ be a nonnegative continuous function and let $b, k \in L^1((0, T))$ be nonnegative functions satisfying*

$$z(t) \leq \gamma + \int_0^t b(s)z(s) \, ds + \int_0^t k(s)z(s)^p \, ds, \quad t \in [0, T],$$

where $p \in (0, 1)$ and $\gamma \geq 0$ are given constants. Then, for all $t \in [0, T]$

$$z(t) \leq \exp\left(\int_0^t b(s) \, ds\right) \left[\gamma^{1-p} + (1-p) \int_0^t k(s) \exp\left((p-1) \int_0^s b(\sigma) \, d\sigma\right) \, ds\right]^{\frac{1}{1-p}}.$$

From this lemma and (4.5) we deduce the fundamental estimate holding true for all $\tau \in [0, T]$:

$$\begin{aligned} z_\varepsilon(\tau) &= \|v_\varepsilon(\tau, \cdot, \cdot)\|_{L^2_\mathbb{C}(\Omega \times \mathcal{O})}^2 + a_0 \|\nabla_x v_\varepsilon\|_{L^2_\mathbb{C}(Q_\tau)}^2 \\ &\leq \left(M_2(\varepsilon, T)^{1/2} \|\tilde{g}\|_{L^2_\mathbb{C}(Q_T)} e^{\beta\tau/2} + \int_0^\tau e^{\beta(\tau-s)/2} \|\tilde{g}_\varepsilon(s, \cdot, \cdot)\|_{L^2_\mathbb{C}(\Omega \times \mathcal{O})} \, ds \right)^2 \\ &\leq 2M_2(\varepsilon, T) \|\tilde{g}\|_{L^2_\mathbb{C}(Q_T)}^2 e^{\beta\tau} + 2 \left(\int_0^\tau e^{\beta(\tau-s)/2} \|\tilde{g}(s, \cdot, \cdot)\|_{L^2_\mathbb{C}(\Omega \times \mathcal{O})} \, ds \right)^2 \\ &\leq 2(M_2(\varepsilon, T) + \beta^{-1}) \|\tilde{g}\|_{L^2_\mathbb{C}(Q_T)}^2 e^{\beta\tau}, \end{aligned}$$

for any $\tau \in [0, T]$. In particular, for all $\tau \in [2\varepsilon T, T]$ we find the following estimate for v , where we have set $Q(2\varepsilon T, \tau) = (2\varepsilon T, \tau) \times \Omega \times \mathcal{O}$:

$$\|v(\tau, \cdot, \cdot)\|_{L^2_\mathbb{C}(\Omega \times \mathcal{O})}^2 + a_0 \|\nabla_x v\|_{L^2_\mathbb{C}(Q(2\varepsilon T, \tau))}^2 \leq 2[M_2(\varepsilon, T) + \beta^{-1}] \|\tilde{g}\|_{L^2_\mathbb{C}(Q_T)}^2 e^{\beta\tau}, \quad (4.6)$$

for any $\tau \in [0, T]$. Recalling that the solution u to problem (2.1) is related to v by the formula $u = v + h$, from (4.6) we immediately deduce the estimate for u :

$$\begin{aligned} &\|u(\tau, \cdot, \cdot)\|_{L^2_\mathbb{C}(\Omega \times \mathcal{O})}^2 + a_0 \|\nabla_x u\|_{L^2_\mathbb{C}(Q(2\varepsilon T, \tau))}^2 \\ &\leq 2\|h(\tau, \cdot, \cdot)\|_{L^2_\mathbb{C}(\Omega \times \mathcal{O})}^2 + 2a_0 \|\nabla_x h\|_{L^2_\mathbb{C}(Q(2\varepsilon T, \tau))}^2 + 4[M_2(\varepsilon, T) + \beta^{-1}] \|\tilde{g}\|_{L^2_\mathbb{C}(Q_T)}^2 e^{\beta\tau}, \end{aligned} \quad (4.7)$$

for any $\tau \in [2\varepsilon T, T]$. Now, taking advantage of definition (2.3), we can estimate

$$\begin{aligned} \|\tilde{g}\|_{L^2_\mathbb{C}(Q_T)}^2 &\leq 6\|g\|_{L^2_\mathbb{C}(Q_T)}^2 + 6\|D_t h\|_{L^2_\mathbb{C}(Q_T)}^2 + 6\|\operatorname{div}_x(a \nabla_x h)\|_{L^2_\mathbb{C}(Q_T)}^2 \\ &\quad + 6\|c\|_\infty^2 \|\nabla_x h\|_{L^2_\mathbb{C}(Q_T)}^2 + 6\|b \cdot \nabla_y h\|_{L^2_\mathbb{C}(Q_T)}^2 + 6\|b_0\|_\infty^2 \|h\|_{L^2_\mathbb{C}(Q_T)}^2. \end{aligned} \quad (4.8)$$

Finally, (4.7) and (4.8) yield the following continuous dependence estimate for all $\tau \in [2\varepsilon T, T]$:

$$\begin{aligned} &\|u(\tau, \cdot, \cdot)\|_{L^2_\mathbb{C}(\Omega \times \mathcal{O})}^2 + a_0 \|\nabla_x u\|_{L^2_\mathbb{C}(Q(2\varepsilon T, \tau))}^2 \\ &\leq M_3(\varepsilon, T) \left\{ \|g\|_{L^2_\mathbb{C}(Q_T)}^2 + \|h\|_{H^1((0, T); L^2_\mathbb{C}(\Omega \times \mathcal{O}))}^2 + \|\operatorname{div}_x(a \nabla_x h)\|_{L^2_\mathbb{C}(Q_T)}^2 \right. \\ &\quad \left. + \|b \cdot \nabla_y h\|_{L^2_\mathbb{C}(Q_T)}^2 + \|\nabla_x h\|_{L^2_\mathbb{C}(Q_T)}^2 \right\}, \end{aligned} \quad (4.9)$$

where the positive constant M_3 depends also on a_0 , $\|a\|_{2, \infty}$, $\|b_0\|_\infty$, $\|\operatorname{div} b\|_\infty$, $\|c\|_\infty$, $\|\psi\|_{3, \infty}$ and α .

We have so proved the following continuous dependence result:

Theorem 4.2. *Under Hypotheses 2.1 the solution u to problem (2.1) satisfies the continuous dependence estimate (4.9).*

5. SOME EXTENSIONS OF OUR MAIN RESULTS

In this section we show that the validity of Theorem 4.2 can be extended both to some classes of degenerate integrodifferential boundary problems and to some classes of semilinear problems.

5.1. A degenerate convolution integrodifferential problem. Here we consider a convolution integrodifferential problem with no initial conditions, and with Cauchy data on the lateral boundary of the cylinder $\Omega \times \mathbb{R}^N$. We still assume that Ω is a bounded subset of \mathbb{R}^M with a boundary of class C^3 .

Let \tilde{A} be the following degenerate integrodifferential linear operator

$$\begin{aligned} \tilde{A}z(x, y) = & \operatorname{div}_x(a(x)\nabla_x z(x, y)) + By \cdot \nabla_y z(x, y) + \beta_1(x) \cdot \nabla_x z(x, y) \\ & + \beta_0(x)z(x, y) + \int_{\mathbb{R}^N} k_1(x, y - \eta) \cdot \nabla_x z(x, \eta) d\eta \\ & + \int_{\mathbb{R}^N} k_0(x, y - \eta)z(x, \eta) d\eta. \end{aligned}$$

Consider the parabolic integrodifferential problem with no initial condition, but with Cauchy data on the boundary

$$\begin{cases} D_t z(t, x, y) = \tilde{A}z(t, x, y) + f(t, x, y), & (t, x, y) \in (0, T) \times \Omega \times \mathbb{R}^N, \\ z(t, x, y) = h(t, x, y), & (t, x, y) \in [0, T] \times \partial\Omega \times \mathbb{R}^N, \\ D_\nu z(t, x, y) = D_\nu h(t, x, y), & (t, x, y) \in [0, T] \times \Gamma \times \mathbb{R}^N, \end{cases} \quad (5.1)$$

where Γ is an open subset of $\partial\Omega$ and

Hypothesis 5.1. *The following conditions are satisfied:*

- (i) $a \in W^{2,\infty}(\Omega)$ and there exists a positive constant a_0 such that $|a(x)| \geq a_0$ for any $x \in \Omega$;
- (ii) B is a real $(N \times N)$ -square matrix;
- (iii) $\beta_0 \in L^\infty(\Omega)$;
- (iv) $\beta_1 \in (L^\infty(\Omega))^M$;
- (v) $k_0 \in L^\infty(\Omega; L^1(\mathbb{R}^N))$;
- (vi) $k_1 \in L^\infty(\Omega; L^1(\mathbb{R}^N))^M$;
- (vii) $f \in L^2(Q_T)$;
- (viii) $h \in H^1((0, T); L^2(\Omega \times \mathbb{O})) \cap L^2((0, T); \mathcal{H}^2(\Omega \times \mathbb{O}))$.

Denote by \mathcal{F}_y the Fourier transform with respect to the variable y . As it is easily seen, function $u = \mathcal{F}_y z$ solves the ill-posed problem

$$\begin{cases} D_t u(t, x, \eta) - \operatorname{div}_x(a(x)\nabla_x u(t, x, \eta)) + c(x, \eta) \cdot \nabla_x u(t, x, \eta) \\ + B^T \eta \cdot \nabla_\eta u(t, x, \eta) + b_0(x, \eta)u(t, x, \eta) = (\mathcal{F}_y f)(t, x, \eta), & (t, x, \eta) \in (0, T) \times \Omega \times \mathbb{R}^N, \\ u(t, x, \eta) = (\mathcal{F}_y h)(t, x, \eta), & (t, x, \eta) \in [0, T] \times \partial\Omega \times \mathbb{R}^N, \\ D_\nu u(t, x, \eta) = (\mathcal{F}_y D_\nu h)(t, x, \eta), & (t, x, \eta) \in [0, T] \times \Gamma \times \mathbb{R}^N, \end{cases}$$

where $c : \Omega \times \mathbb{R}^N \rightarrow \mathbb{C}^M$ and $\beta_0 : \Omega \times \mathbb{R}^N \rightarrow \mathbb{C}$ are defined by

$$c(x, \eta) = -\beta_1(x) - (\mathcal{F}_y k_1)(x, \eta), \quad b_0(x, \eta) = \operatorname{Tr}(B) - \beta_0(x) - (\mathcal{F}_y k_0)(x, \eta),$$

for any $x \in \Omega$ and $\eta \in \mathbb{R}^N$. By Theorem 4.2, u satisfies the continuous dependence estimate

$$\begin{aligned} & \|u(\tau, \cdot, \cdot)\|_{L^2(\Omega \times \mathbb{O})}^2 + a_0 \|\nabla_x u\|_{L^2(Q(2\varepsilon T, \tau))}^2 \\ & \leq M(\varepsilon, T) \left\{ \|\mathcal{F}_y f\|_{L^2(Q_T)}^2 + \|\mathcal{F}_y h\|_{H^1((0, T); L^2(\Omega \times \mathbb{O}))}^2 + \|\operatorname{div}_x(a \nabla_x \mathcal{F}_y h)\|_{L^2(Q_T)}^2 \right\} \end{aligned}$$

$$+ \|B^T \eta \cdot \nabla_\eta \mathcal{F}_y h\|_{L^2(Q_T)}^2 + \|\nabla_x \mathcal{F}_y h\|_{L^2(Q_T)}^2 \Big\}, \quad (5.2)$$

for all $\varepsilon \in (0, 1/4)$, $\tau \in [2\varepsilon T, T]$ and some positive constant $M(\varepsilon, T)$, depending also on a_0 , $\|a\|_{2,\infty}$, $\|B\|$, $\|\beta_0\|_\infty$, $\|\beta_1\|_\infty$, $\|\psi\|_{3,\infty}$, $\|k_0\|_{L^\infty(\Omega; L^1(\mathbb{R}^N))}$, $\|k_1\|_{L^\infty(\Omega; L^1(\mathbb{R}^N))^M}$ and α .

Using the Parseval identity and observing that ∇_x commutes with \mathcal{F}_y and that φ_{ρ_0} is independent of η , from (5.2) we deduce that z satisfies

$$\begin{aligned} & \|z(\tau, \cdot, \cdot)\|_{L^2(\Omega \times \mathcal{O})}^2 + a_0 \|\nabla_x z\|_{L^2(Q(2\varepsilon T, \tau))}^2 \\ & \leq M(\varepsilon, T) \Big\{ \|f\|_{L^2(Q_T)}^2 + \|h\|_{H^1((0,T); L^2(\Omega \times \mathcal{O}))}^2 + \|\operatorname{div}_x(a \nabla_x h)\|_{L^2(Q_T)}^2 \\ & \quad + \|B y \cdot \nabla_y h\|_{L^2(Q_T)}^2 + \|\nabla_x h\|_{L^2(Q_T)}^2 \Big\}, \end{aligned} \quad (5.3)$$

for all $\varepsilon \in (0, 1/4)$ and $\tau \in [2\varepsilon T, T]$.

We have proved the following continuous dependence result:

Theorem 5.2. *Let Hypotheses 5.1 be satisfied. Then, the solution z to problem (5.1) satisfies the continuous dependence estimate (5.3). In particular, if $(\beta_0, \beta_1, f, h) = (0, 0, 0, 0)$, then $z = 0$ in Q_T , i.e., the unique continuation property holds true.*

5.2. A semilinear parabolic equation. We now consider the following semilinear boundary value problem

$$\begin{cases} D_t u(t, x, y) = \operatorname{div}_x(a(x) \nabla_x u(t, x, y)) + b(y) \cdot \nabla_y u(t, x, y) \\ \quad + q(u(t, x, y), \nabla_x u(t, x, y)) + g(t, x, y), & (t, x, y) \in [0, T] \times \Omega \times \mathcal{O} =: Q_T, \\ u(t, x, y) = h(t, x, y), & (t, x, y) \in [0, T] \times \partial_*(\Omega \times \mathcal{O}), \\ D_\nu u(t, x, y) = D_\nu h(t, x, y), & (t, x, y) \in [0, T] \times \Gamma \times \mathcal{O}. \end{cases} \quad (5.4)$$

where Ω and \mathcal{O} and a and b are as in the previous sections (see Hypothesis 2.1), whereas function q satisfies the following condition

Hypothesis 5.3. $q : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ is a Lipschitz-continuous function with a Lipschitz constant κ .

Theorem 5.4. *Let $u_j \in H^1((0, T); L^2_{\mathbb{C}}(\Omega \times \mathcal{O})) \cap L^2((0, T); \mathcal{H}^2_{\mathbb{C}}(\Omega \times \mathcal{O}))$ be a solution to problem (5.4) corresponding to $(g, h) = (g_j, h_j)$, $j = 1, 2$. Then, for any $\varepsilon \in (0, 1/2)$, there exists a positive constant $C = C(\varepsilon, T)$ such that*

$$\begin{aligned} & \|u_2(\tau, \cdot, \cdot) - u_1(\tau, \cdot, \cdot)\|_{L^2_{\mathbb{C}}(\Omega \times \mathcal{O})}^2 + a_0 \|\nabla_x u_2 - \nabla_x u_1\|_{L^2_{\mathbb{C}}(Q(2\varepsilon T, \tau))}^2 \\ & \leq C \Big\{ \|g_2 - g_1\|_{L^2_{\mathbb{C}}(Q_T)}^2 + \|h_2 - h_1\|_{H^1((0,T); L^2_{\mathbb{C}}(\Omega \times \mathcal{O}))}^2 + \|\operatorname{div}_x(a \nabla_x h_2 - a \nabla_x h_1)\|_{L^2_{\mathbb{C}}(Q_T)}^2 \\ & \quad + \|b \cdot \nabla_y h_2 - b \cdot \nabla_y h_1\|_{L^2_{\mathbb{C}}(Q_T)}^2 + \|\nabla_x h_2 - \nabla_x h_1\|_{L^2_{\mathbb{C}}(Q_T)}^2 \Big\}, \end{aligned} \quad (5.5)$$

for any $\tau \in [2\varepsilon T, T]$.

Proof. The proof follows adapting the arguments in Sections 3 and 4. Hence, we just point out the differences.

Note that, if we set

$$\mathcal{P}(u) = D_t u - \operatorname{div}_x(a \nabla_x u) - b \cdot \nabla_y u - q(u, \nabla_x u) =: \mathcal{P}_0(u) - q(u, \nabla_x u),$$

we can rewrite the differential equation in (5.4) in the much more compact form: $\mathcal{P}u = g$.

First we perform the translations $v_j = u_j - h_j$, $j = 1, 2$, and observe that the function $v = v_2 - v_1$ solves problem (5.4) with $q(v, \nabla_x v)$ and (g, h) being replaced,

respectively, by $Q(v_1, v_2)$ and $(g_2 - g_1, 0)$, where

$$Q(v_1, v_2) = q(v_2 + h_2, \nabla_x v_2 + \nabla_x h_2) - q(v_1 + h_1, \nabla_x v_1 + \nabla_x h_1) \quad (5.6)$$

and

$$\tilde{g}_j = g - D_t h_j + \operatorname{div}_x(a \nabla_x h_j) + b \cdot \nabla_y h_j, \quad j = 1, 2. \quad (5.7)$$

Moreover,

$$\mathcal{P}(v) = \mathcal{P}_0(v) - Q(v_1, v_2). \quad (5.8)$$

Since q is a Lipschitz continuous function in \mathbb{C}^{n+1} , we can estimate (pointwise)

$$|Q(v_1, v_2)| \leq h_q(|v_2 - v_1| + |\nabla_x(v_2 - v_1)|) + h_q(|h_2 - h_1| + |\nabla_x(h_2 - h_1)|). \quad (5.9)$$

From the Carleman estimate in Theorem 3.5, (5.8) and (5.9), we can infer that v satisfies the following integral inequality for all $\lambda \geq \lambda_0$:

$$\begin{aligned} & \int_{Q_T} (\lambda |\nabla_x \varphi_{\rho_0}| |\nabla_x v|^2 + \lambda^3 |\nabla_x \varphi_{\rho_0}|^3 |v|^2) e^{2\lambda \varphi_{\rho_0}} dt dx dy \\ & \leq \frac{32}{3} \int_{Q_T} |\mathcal{P}_0 v|^2 e^{2\lambda \varphi_{\rho_0}} dt dx dy \\ & \leq \frac{160}{3} \int_{Q_T} |\mathcal{P}(v_2) - \mathcal{P}(v_1)|^2 e^{2\lambda \varphi_{\rho_0}} dt dx dy \\ & \quad + \frac{160}{3} \kappa^2 \int_{Q_T} [|v_2 - v_1|^2 + |\nabla_x(v_2 - v_1)|^2] e^{2\lambda \varphi_{\rho_0}} dt dx dy \\ & \quad + \frac{160}{3} \kappa^2 \int_{Q_T} [|h_2 - h_1|^2 + |\nabla_x(h_2 - h_1)|^2] e^{2\lambda \varphi_{\rho_0}} dt dx dy \\ & \leq \frac{160}{3} \int_{Q_T} |\mathcal{P}(v_2) - \mathcal{P}(v_1)|^2 e^{2\lambda \varphi_{\rho_0}} dt dx dy \\ & \quad + \frac{160}{3} \kappa^2 \int_{Q_T} \left\{ \left[\left(\frac{\ell}{\alpha \rho_0} \right)^3 |\nabla_x \varphi_{\rho_0}|^3 |(v_2 - v_1)(t, x, y)|^2 \right. \right. \\ & \quad \left. \left. + \frac{\ell}{\alpha \rho_0} |\nabla_x \varphi_{\rho_0}| |\nabla_x(v_2 - v_1)|^2 \right] e^{2\lambda \varphi_{\rho_0}} \right\} dt dx dy \\ & \quad + \frac{160}{3} \kappa^2 \int_{Q_T} [|h_2 - h_1|^2 + |\nabla_x(h_2 - h_1)|^2] e^{2\lambda \varphi_{\rho_0}} dt dx dy. \end{aligned}$$

Whence we deduce the Carleman estimate for v :

$$\begin{aligned} & \int_{Q_T} (\lambda |\nabla_x \varphi_{\rho_0}| |\nabla_x(v_2 - v_1)|^2 + \lambda^3 |\nabla_x \varphi_{\rho_0}|^3 |v_2 - v_1|^2) e^{2\lambda \varphi_{\rho_0}} dt dx dy \\ & \leq \frac{320}{3} \int_{Q_T} |\mathcal{P}(v_2) - \mathcal{P}(v_1)|^2 e^{2\lambda \varphi_{\rho_0}} dt dx dy, \\ & \quad + \frac{320}{3} \kappa^2 \int_{Q_T} [|h_2 - h_1|^2 + |\nabla_x(h_2 - h_1)|^2] e^{2\lambda \varphi_{\rho_0}} dt dx dy, \end{aligned}$$

if we choose

$$\lambda \geq \max \left\{ \lambda_0, \left(\frac{320}{3} \kappa^2 \right)^{1/3} \frac{\ell}{\alpha \rho_0}, \frac{320}{3} \kappa^2 \frac{\ell}{\alpha \rho_0} \right\}.$$

Now, we are almost done. Indeed, arguing as in the proof of (4.2) we can show that

$$D_t \|v_\varepsilon(t, \cdot, \cdot)\|_{L^2_{\mathbb{C}}(\Omega \times \mathcal{O})}^2 + a_0 \|\nabla_x v_\varepsilon(t, \cdot, \cdot)\|_{L^2_{\mathbb{C}}(\Omega \times \mathcal{O})}^2$$

$$\begin{aligned}
&\leq \|\operatorname{div} b\|_\infty \|v_\varepsilon(t, \cdot, \cdot)\|_{L^2_{\mathbb{C}}(\Omega \times \mathcal{O})}^2 + a_0^{-1} \int_{\Omega \times \mathcal{O}} \sigma_\varepsilon(t) |Q(v_1, v_2)| |v_\varepsilon| dx dy \\
&\quad + 2 \|\tilde{g}_{2,\varepsilon}(t, \cdot, \cdot) - \tilde{g}_{1,\varepsilon}(t, \cdot, \cdot)\|_{L^2_{\mathbb{C}}(\Omega \times \mathcal{O})} \|v_\varepsilon(t, \cdot, \cdot)\|_{L^2_{\mathbb{C}}(\Omega \times \mathcal{O})} \\
&\quad + 2 |\sigma'_\varepsilon(t)| \|v(t, \cdot, \cdot)\|_{L^2_{\mathbb{C}}(\Omega \times \mathcal{O})}^2,
\end{aligned}$$

for any $t \in (0, T)$, where $g_{j,\varepsilon} = \sigma_\varepsilon g_j$ ($j = 1, 2$) and σ_ε is given by (4.1). Since q is Lipschitz continuous, we can estimate

$$\begin{aligned}
|v_\varepsilon| |\sigma_\varepsilon Q(v_1, v_2)| &\leq \kappa |v_\varepsilon|^2 + \kappa |v_\varepsilon| |\nabla_x v_\varepsilon| + \kappa |v_\varepsilon| |h_2 - h_1| + \kappa |v_\varepsilon| |\nabla_x h_2 - \nabla_x h_1| \\
&\leq 2\kappa |v_\varepsilon|^2 + \kappa |\nabla_x v_\varepsilon|^2 + \kappa |h_2 - h_1|^2 + \kappa |\nabla_x h_2 - \nabla_x h_1|^2 \quad (5.10)
\end{aligned}$$

and, consequently,

$$\begin{aligned}
&D_t \|v_\varepsilon(t, \cdot, \cdot)\|_{L^2_{\mathbb{C}}(\Omega \times \mathcal{O})}^2 + a_0 \|\nabla_x v_\varepsilon(t, \cdot, \cdot)\|_{L^2_{\mathbb{C}}(\Omega \times \mathcal{O})}^2 \\
&\leq \|\operatorname{div} b\|_\infty \|v_\varepsilon(t, \cdot, \cdot)\|_{L^2_{\mathbb{C}}(\Omega \times \mathcal{O})}^2 + 2a_0^{-1} \kappa \|v_\varepsilon(t, \cdot, \cdot)\|_{L^2_{\mathbb{C}}(\Omega \times \mathcal{O})}^2 \\
&\quad + a_0^{-1} \kappa \|\nabla_x v(t, \cdot, \cdot)\|_{L^2_{\mathbb{C}}(\Omega \times \mathcal{O})}^2 + \frac{1}{2} a_0^{-1} \kappa \|h_2(t, \cdot, \cdot) - h_1(t, \cdot, \cdot)\|_{L^2_{\mathbb{C}}(\Omega \times \mathcal{O})}^2 \\
&\quad + \frac{1}{2} a_0^{-1} \kappa \|\nabla_x h_2(t, \cdot, \cdot) - \nabla_x h_1(t, \cdot, \cdot)\|_{L^2_{\mathbb{C}}(\Omega \times \mathcal{O})}^2 \\
&\quad + 2 \|\tilde{g}_{2,\varepsilon}(t, \cdot, \cdot) - \tilde{g}_{1,\varepsilon}(t, \cdot, \cdot)\|_{L^2_{\mathbb{C}}(\Omega \times \mathcal{O})} \|v_\varepsilon(t, \cdot, \cdot)\|_{L^2_{\mathbb{C}}(\Omega \times \mathcal{O})} \\
&\quad + 2 |\sigma'_\varepsilon(t)| \|v(t, \cdot, \cdot)\|_{L^2_{\mathbb{C}}(\Omega \times \mathcal{O})}^2.
\end{aligned}$$

From (5.10), reasoning as in the previous section, we easily deduce the desired continuity estimates (4.6) for v , where $\tilde{g} = \tilde{g}_2 - \tilde{g}_1$.

Now, the proof follows the same lines as the proof of Theorem 4.2 and yields (5.5). \square

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